Practice Exam 1 Solutions

Problem 1. Let A be an $m \times n$ matrix and r be the rank of A.

1. Describe the dimension of the solution space of the equation $A\mathbf{x} = \mathbf{0}$ in terms of m, n, r.
2. Suppose there exists $\mathbf{c}$ such that $A\mathbf{x} = \mathbf{c}$ does not have a solution. What can you say about m, n, r?
3. If A is invertible, what is the relationship between m, n and r?

Solution

1. Since $A : \mathbb{R}^n \to \mathbb{R}^m$ and $rank(A) = r$ the rank-nullity theorem implies $dim(N(A)) = n - r$.
2. The statement implies that $dim(A(\mathbb{R}^n)) \neq m$. That is, $rank(A) \neq m$. Thus, $r < m$.
3. If A is invertible then $m = n = r$. This follows as only square matrices are invertible, and any invertible matrix must have full rank.

Problem 2. Let $\{x_1, x_2, \cdots, x_n\}$ be a basis for the vector space V. Consider the set $\{\sum_{i=1}^{n} c_{1i}x_i, \cdots, \sum_{i=1}^{n} c_{ni}x_i\}$ for $c_{ij} \in \mathbb{R}$. Is this still a basis for V? Prove it either way.

Solution The new set may or may not be a basis, and depends entirely on the coefficients c_{ji} (as it should). We will show that the new set is a basis if and only if the matrix C corresponding to the entries c_{ji} is invertible.

Suppose there exist $d_j \in \mathbb{R}$ for $j = 1, \ldots, n$ such that

$$\sum_{j=1}^{n} d_j \left(\sum_{i=1}^{n} c_{ji} x_i \right) = 0.$$

Then immediately we have

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} d_j c_{ji} \right) x_i = 0$$

and thus by the independence of the x_i, $\sum_{j=1}^{n} d_j c_{ji} = 0$ for each i. The independence of the new set follows if and only if this implies $d_j = 0$ for each j. Notice that the situation is reduced to solving a system of n equations with n variables. In fact, if C is the matrix such that c_{ji} is the entry in the i^{th} row and j^{th} column, then we wish to solve the system

$$C\mathbf{d} = \mathbf{0}$$

where $\mathbf{d} \in \mathbb{R}^n$. Obviously there exist $\mathbf{d} \neq \mathbf{0}$ exactly when $rank(C) < n$, or when C is not invertible. Thus, the new set is a basis precisely when C is invertible.

Problem 3: Let A, B and C be three vectors (or points) in $\mathbb{R}^3$. Let M be the 3×3 matrix that has A , B and C as its rows (from top to bottom).

1. Show that $|\det M| \leq ||A||||B||||C||$.
2. Show that if $\{A, B, C\}$ is an orthogonal set then $\det M = \pm||A||||B||||C||$. When does one get a $+$ and when a $-$?
3. Is it true that if $|\det M| = ||A||||B||||C||$ then $\{A, B, C\}$ is orthogonal?

Solution For all three parts, we use the fact that $A \cdot (B \times C) = det(A, B, C)$. In fact

$$A \cdot (B \times C) = ||A||||B \times C|| \cos\theta = ||A||||B||||C|| \cos\theta \sin\phi$$

where $0 \leq \theta \leq \pi$ is the angle between A and $B \times C$ and $0 \leq \phi \leq \pi$ is the angle between B and C. Notice the absolute value is maximized precisely when $\theta = 0, \pi$ and $\phi = \pi/2$.

1. Follows immediately from work above.
2. First, notice $\sin\phi = 1$ iff B is orthogonal to C. Now, $|\cos\theta| = 1$ iff $B \times C$ and A are parallel; moreover $\cos\theta = \pm 1$ when $A = \pm\lambda B \times C$ for $\lambda \in \mathbb{R}^+$. That is, $\cos\theta = 1$ if $B \times C$ points in the same direction as A and is -1 in the other case. So the $+$ comes when $\{A, B, C\}$ is an ordered orthonormal set and the $-$ comes when $\{A, C, B\}$ is an ordered orthonormal set.
3. This is obviously true based on the work outlined above.

Problem 4. Let L be a map from $\mathbb{R}^3$ to $\mathbb{R}^2$ for which

$$L(u + v) = L(u) + L(v) \qquad (u, v \in \mathbb{R}^3).$$

1. Show that $L(nv) = nL(v)$ for any integer n and $v \in \mathbb{R}^3$;

2. Show that $L(\frac{1}{n}v) = \frac{1}{n}L(v)$ for any integer n and $v \in \mathbb{R}^3$;

3. Show that $L(\frac{m}{n}v) = \frac{n}{m}L(v)$ for any rational number $\frac{n}{m}$ and $v \in \mathbb{R}^3$;

4. Conclude that if L is continuous, then L must be linear. (We say L is continuous at y if $||L(x) - L(y)|| \to 0$ when $||x - y|| \to 0$.)

Solution

1. First, we observe that $L(0) = L(0 + 0) = L(0) + L(0) = 2L(0)$ which implies $L(0) = 0$. Moreover, $0 = L(0) = L(x + (-x)) = L(x) + L(-x)$ and thus $L(-x) = -L(x)$ for all $x \in \mathbb{R}^3$.

   We prove the first statement by induction for $n \in \mathbb{Z}^+$ and then use the work above to prove for all $n \in \mathbb{Z}$. As $L(v) = 1 \cdot L(v)$ we begin by assuming $L(nv) = nL(v)$ and prove that $L((n + 1)v) = (n + 1)L(v)$. This is immediate as $L((n + 1)v) = L(nv + v) = L(nv) + L(v)$ by the assumption on L. As $L(nv) = nL(v)$, we get $L((n + 1)v) = nL(v) + L(v) = (n + 1)L(v)$.

2. Consider $L(v) = L(n \cdot \frac{1}{n}v) = nL(\frac{1}{n}v)$ by the work above (for $n \neq 0$). But then $\frac{1}{n}L(v) = L(\frac{1}{n}v)$.

3. First observe that $\frac{n}{m}v = n(\frac{1}{m}v)$. Thus, using the two parts above we see

$$L(\frac{n}{m}v) = L(n\frac{1}{m}v) = nL(\frac{1}{m}v) = n\frac{1}{m}L(v) = \frac{n}{m}L(v).$$

4. Consider any $c \in \mathbb{R}$ and let $\{r_i\}$ be a sequence of rational numbers such that $r_i \to c$. Then $||r_i v - cv|| = |r_i - c|||v||$ and thus $\lim_{i\to\infty} ||r_i v - cv|| = 0$. It follows that

$$cL(v) = \lim_{i\to\infty} r_i L(v) = \lim_{i\to\infty} L(r_i v) = L(cv)$$

   Thus, for all $c \in \mathbb{R}$, $L(cv) = cL(v)$ and thus L is linear.

Problem 5. Consider the function

$$f(x, y) = \begin{cases} (x^2 + y^2)\sin\frac{1}{x^2+y^2} & \text{if } x^2 + y^2 \neq 0, \\ 0 & \text{if } x = y = 0, \end{cases}$$

1. Show that the partial derivatives of f are discontinuous at $(0,0)$;
2. Show that the partial derivatives of f are not bounded in any balls around $(0,0)$;
3. Show that f is differentiable at $(0,0)$.

Solution We solve the first two parts only for the x derivative as the symmetry of the function in x and y will determine the same result for the y derivative. First, we determine $f'(\mathbf{0};\mathbf{e}_1)$. (This is really $\partial f/\partial x(0,0)$.) By definition

$$f'(\mathbf{0};\mathbf{e}_1) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} h^2 \sin\left(\frac{1}{h^2}\right) = 0.$$

Thus

$$\frac{\partial f}{\partial x}(0,0) = 0.$$

Now consider $x' \neq 0, y = 0$, and determine $\frac{\partial f}{\partial x}(x',0)$. By the differentiability of f away from $x = y = 0$ we can simply calculate

$$\frac{\partial f}{\partial x}(x',0) = 2x'\sin(1/x'^2) - \frac{2}{x'}\cos(1/x'^2).$$

Observe that

$$\lim_{x'\to 0} 2x'\sin(1/x'^2) - \frac{2}{x'}\cos(1/x'^2) = -\infty.$$

Proving this in detail requires more work than I give here (though you should be able to do it quite easily!). You simply use the boundedness of the sin, cos functions and the behavior of the linear and rational terms.

And now we answer the first two questions. For any $\delta > 0, M > 0$ there exists $(x',0) \in B_\delta^2(\mathbf{0})$ such that $|\frac{\partial f}{\partial x}(x',0)| > M$. Simply choose x' such that $\cos(1/x'^2) = 1$ and $x' \leq \min\{\delta/4, 2/M\}$. Thus, $\frac{\partial f}{\partial x}$ is unbounded in any ball around $(0,0)$.

Also, we see $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$. This follows since, for $\epsilon = M = 1$ and any $\delta > 0$ there exists $(x',0) \in B_\delta^2(\mathbf{0})$ such that $|\frac{\partial f}{\partial x}(x',0) - \frac{\partial f}{\partial x}(0,0)| = |\frac{\partial f}{\partial x}(x',0)| > 1 = \epsilon$.

Finally, we show that f is actually differentiable at $(0,0)$. It is enough to show there exists a linear transformation T such that

$$\lim_{||(x,y)||\to 0}\frac{f(x,y)-f(0,0)-T(x,y)}{||(x,y)||} = 0.$$

We provide the candidate $T(x,y)=0$ and show it works. (This is the only candidate as $\nabla f(0,0)=0$.) Notice $||(x,y)||^2 = x^2+y^2$. For ease of notation, we denote this value as r^2. Now, $f(x,y)=r^2\sin(1/r^2)$ and thus

$$\frac{f(x,y)-f(0,0)-T(x,y)}{||(x,y)||} = \frac{r^2\sin(1/r^2)}{r} = r\sin(1/r^2).$$

Now $0 \leq |r\sin(1/r^2)| \leq |r|$ for all $r\in\mathbb{R}$. Thus, the squeeze theorem implies $\lim_{r\to 0} r\sin(1/r^2)=0$. This proves f is differentiable at $(0,0)$.

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