Practice Exam 2 Solutions

Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a scalar field. For each of the following questions, answer "yes" or "no". If the answer is "yes", cite a theorem or give a brief sketch of a proof. If the answer is "no", provide a counterexample.

1. Suppose $f'(\mathbf{a};\mathbf{x})$ exists for all $\mathbf{x} \in \mathbb{R}^2$. Is f continuous at $\mathbf{a}$?
2. Suppose $D_1 f, D_2 f$ both exist at $\mathbf{a}$. Does $f'(\mathbf{a};\mathbf{x})$ exist for all $\mathbf{x} \in \mathbb{R}^2$?
3. Suppose f is differentiable at $\mathbf{a}$. Is f continuous at $\mathbf{a}$?
4. Suppose $D_1 f, D_2 f$ both exist at $\mathbf{a}$ and are continuous in a neighborhood of $\mathbf{a}$. Is f continuous at $\mathbf{a}$?

Solution

1. No. See the example on page 257 in the book.
2. No. Define
$$f(x,y) = \begin{cases} 0 \text{ if } x \neq y \text{ or if } (x,y) = (0,0) \\ 1 \text{ if } x = y, x \neq 0 \end{cases}$$
Then $D_1 f = D_2 f = 0$ at the origin. But consider $f'((0,0);(1,1))$. By definition this is
$$\lim_{h\to 0} \frac{f(h,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{1}{h}.$$
Since this blows up, we see $f'((0,0);(1,1))$ does not exist.
3. Yes. We proved this in class. (Also Theorem 8.6 in the book.)
4. Yes. This implies that f is differentiable, and then continuity follows from the previous statement. (See Theorem 8.7 in the book.)

Problem 2. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{f}(x,y) = (x^2 + y, 2x + y^2)$.
Find $D\mathbf{f}$ and determine the values of (x,y) for which $\mathbf{f}$ is NOT invertible.
Given that $\mathbf{f}$ is invertible at $(0,0)$, let $\mathbf{g}$ be its inverse. Find $D\mathbf{g}(0,0)$.

Solution First, we determine

$$D\mathbf{f}(x,y) = \begin{pmatrix} 2x & 1 \\ 2 & 2y \end{pmatrix}$$

So, $det(D\mathbf{f}) = 4xy - 2$ and thus $det(D\mathbf{f}) = 0$ whenever $xy = 1/2$. Second, note

$$D\mathbf{g}(0,0) = (D\mathbf{f}(0,0))^{-1} = \frac{1}{-2}\begin{pmatrix} 2\cdot 0 & -1 \\ -2 & 2\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix}$$

Problem 3: Let $f(x,y,z) = 2x^2y + xy^2z + xyz$ and consider the level surface $f(x,y,z) = 4$.
Find the tangent plane at $(x,y,z) = (1,1,1)$.
Explain why it is possible to find a function $g(x,y)$, defined in a neighborhood of $(x,y) = (1,1)$ such that a neighborhood of $(1,1,1)$ on the surface $f(x,y,z) = 4$ can be written as a graph $(x,y,g(x,y))$.

Solution As $\nabla f(1,1,1) = (4+1+1, 2+2+1, 1+1) = (6,5,2)$, we write the equation for the tangent plane as

$$\nabla f(1,1,1)\cdot(x-1,y-1,z-1) = 0$$

or

$$(6,5,2)(x-1,y-1,z-1) = 6x-6+5y-5+2z-2 = 6x+5y+2z-13 = 0.$$

Since $\frac{\partial f}{\partial z}(1,1,1) \neq 0$, the implicit function theorem states that there exists a function $g(x,y)$ defined in a neighborhood of $(1,1)$ such that $(x,y,g(x,y))$ is a neighborhood of $(1,1,1)$ on the surface $f(x,y,z) - 4 = 0$ (or $f(x,y,z) = 4$).

Problem 4. Find all extreme values for $f(x,y,z) = x^2 + 2y^2 + 4z^2$ subject to the constraint $x + y + z = 7$. Justify whether the extreme values are maximum or a minimum.

Solution We find the extreme values using Lagrange multipliers. Let $g(x,y,z) = x + y + z - 7$. Then there exists a λ such that $\nabla f = \lambda \nabla g$ at any extreme value. Calculating, we see

$$(2x, 4y, 8z) = \lambda(1,1,1)$$

or

$$(x, y, z) = \left(\frac{\lambda}{2}, \frac{\lambda}{4}, \frac{\lambda}{8}\right).$$

Now we use the fact that $x + y + z = 7$ to solve for λ and thus for (x, y, z). That is, the constraint implies

$$\lambda\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{7\lambda}{8} = 7.$$

Given that $\lambda = 8$ we find $(x, y, z) = (4, 2, 1)$ at the only extreme value.

Now observe that as $||(x, y, z)|| \to \infty$, $f(x, y, z) \to \infty$. Therefore, with only one extreme value, $f(4, 2, 1)$ must be a minimum.

Problem 5. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field with $\mathbf{f} = (f_1, f_2, \ldots, f_n)$. We define the divergence of $\mathbf{f}$ such that

$$div(\mathbf{f}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar field. Prove that

$$div(\nabla g) = \sum_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2}.$$

Solution The proof will follow immediately from the definitions. First, observe that

$$\nabla g = (D_1 g, D_2 g, \ldots, D_n g).$$

Now by definition,

$$div(\nabla g) = \sum_{i=1}^{n} \frac{\partial D_i g}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2}.$$

Problem 6: Assume f, g are integrable on the rectangle $Q \subset \mathbb{R}^2$ and let $a, b \in \mathbb{R}$. Given the linearity of the integral for step functions, prove $\int\int_Q (af + bg)dxdy = a\int\int_Q f dxdy + b\int\int_Q g dxdy$.

Solution We first prove that $af + bg$ is integrable on Q and then determine the value of this integral.

By the Riemann condition, for any $\epsilon > 0$ there exist step functions s_f, s_g, t_f, t_g with $s_f \leq f \leq t_f, s_g \leq g \leq t_g$ and

$$\int\int_Q (t_f - s_f) < \epsilon/(2a), \qquad \int\int_Q (t_g - s_g) < \epsilon/(2b).$$

As expected, let $s = as_f + bs_g$ and $t = at_f + bt_g$. Immediately we have $s \leq af + bg \leq t$ for all $\mathbf{x} \in Q$. Also, the linearity of the double integral for step functions implies

$$\int\int_Q (t - s) = \int\int_Q (a(t_f - s_f) + b(t_g - s_g)) = a\int\int_Q t_f - s_f + b\int\int_Q t_g - s_g < a\frac{\epsilon}{2a} + b\frac{\epsilon}{2b} = \epsilon.$$

So the Riemann condition implies $af + bg$ is integrable.

Now the value of $\int\int_Q (af + bg)dxdy$ is determined to be the real number A such that for $\epsilon > 0$ and step functions s, t with $s \leq af + bg \leq t$ and $\int\int_Q t - s < \epsilon$,

$$\int\int_Q s \leq A \leq \int\int_Q t.$$

But note that

$$\int\int_Q s = a\int\int_Q s_f + b\int\int_Q s_g \leq a\int\int_Q f + b\int\int_Q g \leq a\int\int_Q t_f + b\int\int_Q t_g = \int\int_Q t.$$

Thus $A = a\int\int_Q f + b\int\int_Q g$ which gives the result.

18.024 Multivariable Calculus with Theory
Spring 2011