Second derivative test

1. Find and classify all the critical points of

$$f(x, y) = x^6 + y^3 + 6x - 12y + 7.$$

Answer: Taking the first partials and setting them to 0:

$$\frac{\partial z}{\partial x} = 6x^5 + 6 = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = 3y^2 - 12 = 0.$$

The first equation implies $x = -1$ and the second implies $y = \pm 2$. Thus, the critical points are $(-1, 2)$ and $(-1, -2)$.

Taking second partials:

$$\frac{\partial^2 z}{\partial x^2} = 30x^4, \quad \frac{\partial^2 z}{\partial xy} = 0, \quad \frac{\partial^2 z}{\partial y^2} = 6y.$$

We analyze each critical point in turn.
At (-1,-2): $A = z_{xx}(-1, -2) = 30, \quad B = z_{xy}(-1, -2) = 0, \quad C = z_{yy}(-1, -2) = -12.$
Therefore $AC - B^2 = -360 < 0$, which implies the critical point is a saddle.

At (-1,2): $A = z_{xx}(-1, 2) = 30, \quad B = z_{xy}(-1, 2) = 0, \quad C = z_{yy}(-1, 2) = 12.$
Therefore $AC - B^2 = 360 > 0$ and $A > 0$, which implies the critical point is a minimum.