18.02 Problem Set 6, Part II Solutions

Problem 1 (a) $f(x,y) = x^2 - y^2$, $\vec{\nabla} f = 2\langle x, -y\rangle$, $g(x,y) = x^2 + y^2$, $\vec{\nabla} g = 2\langle x, y\rangle$. $\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \langle x, -y\rangle = \lambda\langle x, y\rangle$, or $x = \lambda x$, $y = -\lambda y$.
Two possibilities: $x \neq 0 \to \lambda = 1 \to y = 0$; $y \neq 0 \to \lambda = -1 \to x = 0$. So $\vec{\nabla} f = \vec{\nabla} g$ for all non-zero points on the x-axis ($\lambda = 1$) and $\vec{\nabla} f = -\vec{\nabla} g$ for all non-zero points on the y-axis ($\lambda = -1$).

(b) $g(x,y) = x^2 + y^2 = 3$
$y = 0$, $x = \pm\sqrt{3} \approx 1.732$, $(\pm\sqrt{3}, 0) \approx (\pm 1.73, 0)$ $\lambda = 1$.
$x = 0$, $y = \pm\sqrt{3}$, $(0, \pm\sqrt{3}) \approx (0, \pm 1.732)$ $\lambda = -1$.

(c) $\lambda = +1$ x-axis contact points $f = x^2 - y^2 = 3$ $(y = 0)$ $x = \pm\sqrt{3}$, the two gradients point in the same direction ($\lambda > 0$).

$\lambda = -1$ y-axis contact points $f = x^2 - y^2 = -3$ $(x = 0)$ $y = \pm\sqrt{3}$, the two gradients point in the opposite direction ($\lambda < 0$).

Problem 2
a) We want to minimize

$$I_1^2 R_1 + I_2^2 R_2$$

subject to

$$I_1 + I_2 = I$$

where I is a constant. Using Lagrange multipliers we get the equations:

$$2I_1R_1 = \lambda, \quad 2I_2R_2 = \lambda, \quad I_1 + I_2 = I$$

which we solve to get that

$$I_1 = \frac{R_2}{R_1 + R_2} I, \qquad I_2 = \frac{R_1}{R_1 + R_2} I$$

(If you are familiar with circuits, note that λ is none other than the voltage!)
b) We want to minimize

$$I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3$$

subject to

$$I_1 + I_2 + I_3 = I$$

where I is a constant. Using Lagrange multipliers we get the equations:

$$2I_1R_1 = \lambda, \quad 2I_2R_2 = \lambda, \quad 2I_3R_3 = \lambda, \quad I_1 + I_2 + I_3 = I$$

which we solve to get that

$$I_1 = \frac{R_2R_3}{D}\,I, \quad I_2 = \frac{R_1R_3}{D}\,I, \quad I_3 = \frac{R_2R_1}{D}\,I,$$

where $D = R_1R_3 + R_2R_3 + R_1R_2$

Problem 3

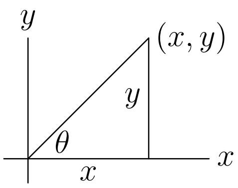

Fig. 1

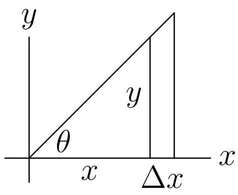

Fig. 2

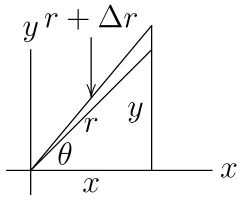

Fig. 3

a) $y = x\tan\theta$ (see Fig. 1). Area $= w = \frac{1}{2}xy = \frac{1}{2}x^2\tan\theta$.

$$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_\theta = x\tan\theta = y, \text{ and } \left(\frac{\partial w}{\partial \theta}\right)_x = \frac{1}{2}x^2\sec^2\theta.$$

b) As before, $y = x\tan\theta$ and $w_x = \frac{1}{2}y$, $w_y = \frac{1}{2}x$.

$$\left(\frac{\partial w}{\partial x}\right)_\theta = w_x\left(\frac{\partial x}{\partial x}\right)_\theta + w_y\left(\frac{\partial y}{\partial x}\right)_\theta = \frac{1}{2}y + \frac{1}{2}x\tan\theta = \frac{1}{2}y + \frac{1}{2}y = y,$$

$$\left(\frac{\partial w}{\partial \theta}\right)_x = w_x\left(\frac{\partial x}{\partial \theta}\right)_x + w_y\left(\frac{\partial y}{\partial \theta}\right)_x = 0 + \frac{1}{2}x^2\sec^2\theta = \frac{1}{2}x^2\sec^2\theta.$$

c) $dw = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$, $dy = \tan\theta\,dx + x\sec^2\theta\,d\theta$.

Eliminate dy from the equation for dw.

$dw = \frac{1}{2}y\,dx + \frac{1}{2}x(\tan\theta\,dx + x\sec^2\theta\,d\theta) = (\frac{1}{2}y + \frac{1}{2}x\tan\theta)dx + (\frac{1}{2}x^2\sec^2\theta)d\theta$.

$$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_\theta = \frac{1}{2}y + \frac{1}{2}x\tan\theta = y, \text{ and } \left(\frac{\partial w}{\partial \theta}\right)_x = \frac{1}{2}x^2\sec^2\theta.$$

d) If we fix θ and vary x then (see Fig. 2)

$\Delta w =$ area of trapezoidal strip at right $= \Delta x \cdot \frac{1}{2}(y + y + \Delta y) = y\Delta x + \frac{1}{2}\Delta x \cdot \Delta y \approx y\Delta x$.

(We ignore second order terms.) $\Rightarrow \dfrac{\Delta w}{\Delta x} \approx y \Rightarrow \left(\dfrac{\partial w}{\partial x}\right)_\theta = y.$

If we fix x and vary θ then (see Fig. 3) $\Delta w =$ area of thin wedge.

The angle of the wedge is $\Delta\theta$ and $\Delta w = \frac{1}{2}r(r + \Delta r)\sin(\Delta\theta) \approx \frac{1}{2}r(r + \Delta r)\Delta\theta \approx \frac{1}{2}r^2\Delta\theta$.

(Here, we've used $\sin x \approx x$ and then dropped second order terms.)

$$\Rightarrow \frac{\Delta w}{\Delta\theta} \approx \frac{1}{2}r^2 = \frac{1}{2}x^2\sec^2\theta \Rightarrow \left(\frac{\partial w}{\partial \theta}\right)_x = \frac{1}{2}x^2\sec^2\theta.$$