Problems: Non-independent Variables

1. Find the total differential for $w = zxe^y + xe^z + ye^z$.

Answer:

$$\begin{aligned} dw &= ze^y\,dx + zxe^y\,dy + xe^y\,dz + e^z\,dx + xe^z\,dz + e^z\,dy + ye^z\,dz \\ &= (ze^y + e^z)dx + (zxe^y + e^z)dy + (xe^y + xe^z + ye^z)dz. \end{aligned}$$

2. With w as above, suppose we have $x = t$, $y = t^2$ and $z = t^3$. Write dw in terms of dt.

Answer: Here $dx = dt$, $dy = 2t\,dt$ and $dz = 3t^2\,dt$. We do not substitute for x, y and z because it does not greatly simplify the expression for dw and because in practice those values may be given or easily calculated from t.

$$dw = (ze^y + e^z)dt + (zxe^y + e^z)2t\,dt + (xe^y + xe^z + ye^z)3t^2\,dt.$$

3. Now suppose w is as above and $x^2y + y^2x = 1$. Assuming x is the independent variable, find $\frac{\partial w}{\partial x}$.

Answer: The constraint $x^2y + y^2x = 1$ becomes $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$. Solving for dy in terms of x, y and dx we get $dy = \frac{2xy + y^2}{x^2 + 2xy}\,dx$.

Using the equation for dw from (1) gives:

$$\begin{aligned} dw &= (ze^y + e^z)dx + (zxe^y + e^z)dy + (xe^y + xe^z + ye^z)dz \\ &= (0 + e^0)dx + (0 + e^0)\left(\frac{2xy + y^2}{x^2 + 2xy}\,dx\right) + (xe^y + xe^z + ye^z)dz \\ &= dx + \frac{2xy + y^2}{x^2 + 2xy}\,dx + (xe^y + xe^z + ye^z)dz \\ &= \frac{x^2 + 4xy + y^2}{x^2 + 2xy}\,dx + (xe^y + xe^z + ye^z)dz. \end{aligned}$$

Thus, $\frac{\partial w}{\partial x} = \frac{x^2 + 4xy + y^2}{x^2 + 2xy}$.