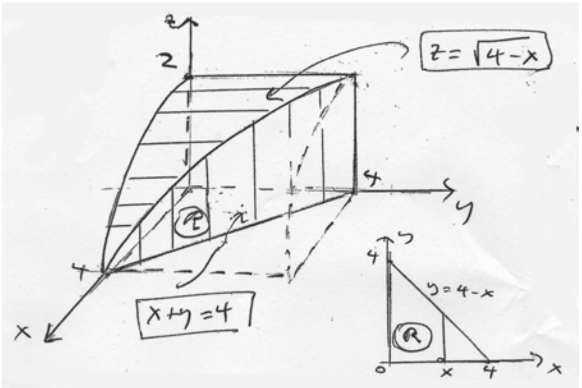
18.02 Problem Set 7, Part II Solutions

1.(a)



(b)

$$\begin{aligned} V &= \int_0^4 \int_0^{4-x} \sqrt{4-x} dy\, dx \\ &= \int_0^4 \left[y\sqrt{4-x}\right]_{y=0}^{y=4-x} dx \\ &= \int_0^4 (4-x)^{3/2} dx = -\frac{2}{5}(4-x)^{5/2}\Big|_0^4 = \frac{2}{5}4^{5/2} = \frac{64}{5}. \end{aligned}$$

2. (a) For simplicity let us assume we are integrating the volume of revolution out to some radius a. We also assume that $f(r) \geq 0$ for $0 \leq r \leq a$. Then if R is the disc $x^2 + y^2 \leq a$, we want

$$V = \int\int_R f dA.$$

In polar coordinates this is

$$V = \int_0^{2\pi} \int_0^a f(r) r dr d\theta.$$

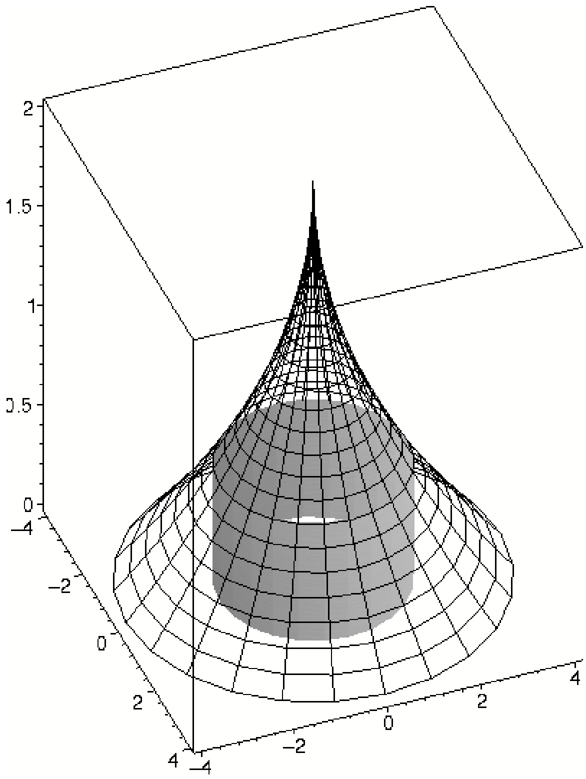
We may write the integral in the other order as well, because the limits to each integral are constants.

$$V = \int_0^a \left(\int_0^{2\pi} f(r) d\theta \right) r dr.$$

Evaluating the inner integral gives

$$\int_0^a 2\pi r f(r) dr$$

which is the shell method formula.



(b)

3. (a) For a circular sector S_θ with center angle 2θ and radius a,

$$A(S_\theta) = \frac{1}{2}a^2(2\theta) = a^2\theta$$

and its centroid is at $(\bar{x}_S(\theta), 0)$ where

$$\bar{x}_S(\theta) = \bar{x}(S_\theta) = \frac{1}{a^2\theta}\int_{-\theta}^{\theta}\int_0^a (r\cos\varphi)\, r\ dr\ d\varphi.$$

This comes out to

$$\bar{x}_S(\theta) \ = \ \left(\frac{2\sin\theta}{3\theta}\right) a.$$

So we observe a factor $f_s(\theta) = \frac{2\sin\theta}{3\theta}$ governing at what multiple of the radius the centroid must occur.

(b) The result from elementary geometry is that the centroid of a triangular region with uniform density is located at the intersection of the three side-bisectors or 'medians', and that this point divides the medians in a ratio of 2 to 1, with the shorter segment nearest the bisection point. Thus we get that for the triangle given and positioned in the same way as the circular sector on the x-axis

$$\bar{x}_\Delta = \left(\frac{2}{3}\right) a.$$

So $f_\Delta =$ the factor which multiplies a is equal to $\frac{2}{3}$, independent of θ.

(c) The circular sector region is a subset of the triangular region, with the excess part of the triangle farther away from the origin. Thus we should have $\bar{x}_S(\theta) < \bar{x}_\Delta$. But in fact the math agrees, since $\frac{\sin\theta}{\theta} < 1$, and so the $f_s(\theta)$, the factor of a for the sector, which we found in part(a) to be $f_s(\theta) = \frac{2\sin\theta}{3\theta}$ thus satisfies the inequality $f_s(\theta) < \frac{2}{3} = f_\Delta$.

4. Case A: $(X(x,y,t), Y(x,y,t)) = ((1+t)x, (1+t)y)$.

$$J(x,y,t) = \frac{\partial(X,Y)}{\partial(x,y)} = \begin{bmatrix} 1+t & 0 \\ 0 & 1+t \end{bmatrix}$$

and so (a) $|\,J(x,y,t)\,| = (1+t)^2$ and (b) $A(\mathcal{R}_t) = (1+t)^2 A(\mathcal{R})$

Case B: $(X(x,y,t), Y(x,y,t)) = (x\cos t - y\sin t, x\sin t + y\cos t)$,

$$J(x,y,t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

and so (a) $|\,J(x,y,t)\,| = 1$ and (b) $A(\mathcal{R}_t) = A(\mathcal{R})$ for all t.

Case C: $(X(x,y,t), Y(x,y,t)) = \big((1+t)x, (\frac{1}{1+t})y\big)$,

$$J(x,y,t) = \begin{bmatrix} 1+t & 0 \\ 0 & \frac{1}{1+t} \end{bmatrix}$$

and so (a) $|\,J(x,y,t)\,| = 1$ and (b) $A(\mathcal{R}_t) = A(\mathcal{R})$ for all t.

5. Case A: $(X(x,y,t), Y(x,y,t)) = ((1+t)x, (1+t)y)$.

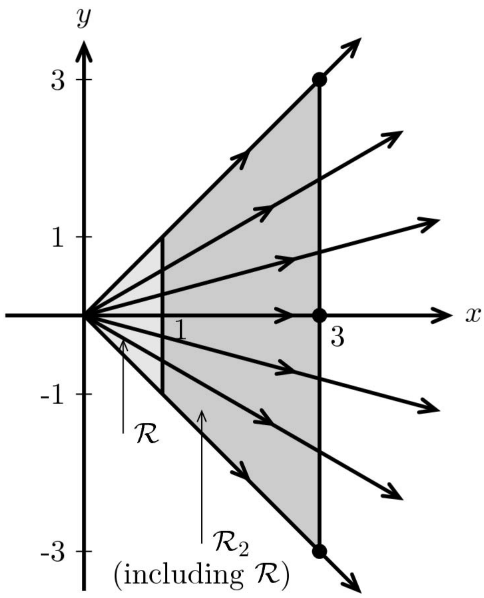
$\mathbf{v}(x,y,t) = \langle \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} \rangle = \langle x, y \rangle$. The flow lines are straight lines fanning out from the origin. The velocity vectors depend only on the position, and their magnitude increases with the distance from the origin; thus the flow gets faster as it moves away from O.

$\mathcal{R}_2$, the points downstream at $t = 2$ from the triangular region $\mathcal{R}$, form a triangular region with vertices at $(0,0)$, $(3,3)$ and $(3,-3)$. Thus $A(\mathcal{R}_2) = \frac{1}{2}\cdot 3\cdot 6 = 9 = (1+2)^2 A(\mathcal{R})$ as predicted in problem 5, since $A(\mathcal{R}) = \frac{1}{2}\cdot 1\cdot 2$

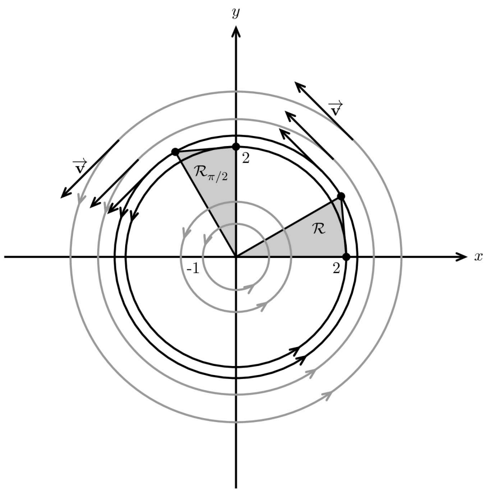
$= 1$. The flow is **not v-i**.

Case B: $(X(x,y,t), Y(x,y,t)) = (x\cos t - y\sin t, x\sin t + y\cos t)$.
$\mathbf{v}(x,y,t) = \langle -x\sin t - y\cos t,\, x\cos t - y\sin t\rangle$. The flow lines are circular paths centered at the origin. The velocity vectors depend on position and time; however the speed $|\,\mathbf{v}(x,y,t)\,| = \sqrt{x^2+y^2}$ does not depend explicitly on time; its magnitude increases with the distance from the origin, but the angular velocity $\omega = \frac{|\mathbf{v}|}{r} = 1$ is constant. So the flow is a 'pure rotating' circular flow moving counter-clockwise around the origin at 1 rad./unit time. $\mathcal{R}_{\frac{\pi}{2}}$, the points downstream at $t = \frac{\pi}{2}$ from the triangular region $\mathcal{R}$, form a triangular region with vertices at $(0,0)$, $(0,2)$ and $(-1,2)$, i.e. the triangular region $\mathcal{R}$ rotated by $\frac{\pi}{2}$ counter-clockwise. Thus $A\left(\mathcal{R}_{\frac{\pi}{2}}\right) = A(\mathcal{R}) = 1$, as predicted in problem 4, since in general the flow **is v-i**.

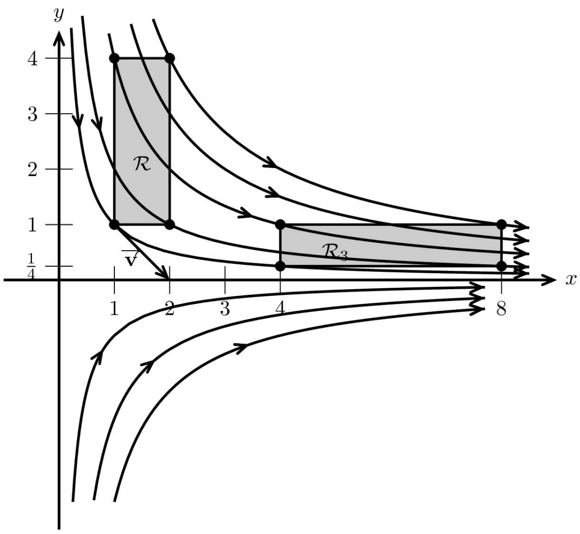
Case C: $(X(x,y,t), Y(x,y,t)) = \left((1+t)x, (\frac{1}{1+t})y\right)$.
$\mathbf{v}(x,y,t) = \langle x, \frac{-y}{(1+t)^2}\rangle$. The flow lines are the hyberbolas $XY = xy =$ constant, with the x and y-axes as asymptotes. The velocity vectors depend on position and time. (The **j**-component of the velocity goes to zero as $t > 0$ increases, which is consistent with the fact that the x-axis is a horizontal asymptote.) The flow comes 'screaming in' at high speed from $(0,\infty)$ for $t > -1$, and then slows down as t increases.
$\mathcal{R}_3$, the points downstream at $t = 3$ from the rectangular region $\mathcal{R}$, form a rectangular region with vertices at $(4,\frac{1}{4})$, $(4,1)$, $(8,\frac{1}{4})$, and $(8,1)$. Thus $A(\mathcal{R}_3) = A(\mathcal{R}) = 3$, as predicted in problem 5, since in general the flow **is v-i**. However, it is not as easy to see why this is the case as it was in case B, where the flow just rotates a region into a congruent region.



y
3
1
-1
-3
x
1
3
$\mathcal{R}$
$\mathcal{R}_2$
(including $\mathcal{R}$)



y
x
$\vec{v}$
$\vec{v}$
2
$\mathcal{R}_{\pi/2}$
$\mathcal{R}$
-1
2



y
4
3
2
1
$\frac{1}{4}$
$\mathcal{R}$
$\mathcal{R}_3$
$\vec{\mathbf{v}}$
1
2
3
4
8
x