Fundamental Theorem for Line Integrals

Gradient fields and potential functions

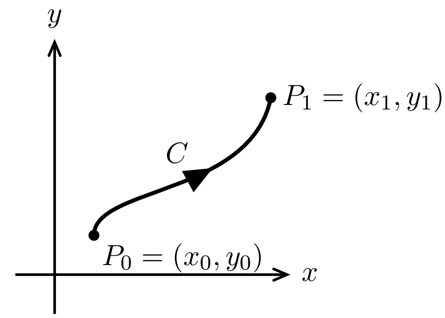
Earlier we learned about the gradient of a scalar valued function

$$\nabla f(x,y) = \langle f_x, f_y \rangle.$$

For example, $\nabla x^3y^4 = \langle 3x^2y^4, 4x^3y^3 \rangle$.

Now that we know about vector fields, we recognize this as a special case. We will call it a *gradient field*. The function f will be called a *potential function* for the field.

For gradient fields we get the following theorem, which you should recognize as being similar to the fundamental theorem of calculus.



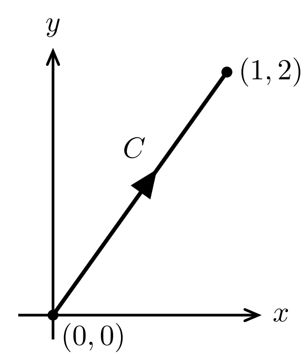
Theorem (Fundamental Theorem for line integrals)
If $\mathbf{F} = \nabla f$ is a gradient field and C is *any* curve with endpoints $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(x,y)|_{P_0}^{P_1} = f(x_1,y_1) - f(x_0,y_0).$$

That is, for *gradient fields* the line integral is independent of the path taken, i.e., it depends only on the endpoints of C.

Example 1: Let $f(x,y) = xy^3 + x^2 \;\Rightarrow\; \mathbf{F} = \nabla f = \langle y^3 + 2x, 3xy^2 \rangle$

Let C be the curve shown and compute $I = \int_C \mathbf{F} \cdot d\mathbf{r}$.



Do this both directly (as in the previous topic) and using the above formula.
Method 1: parametrize C: $x = x$, $y = 2x$, $0 \le x \le 1$.

$$\Rightarrow I = \int_C (y^3 + 2x)\,dx + 3xy^2\,dy = \int_0^1 (8x^3 + 2x)\,dx + 12x^3 2\,dx$$

$$= \int_0^1 32x^3 + 2x\,dx = 9.$$

Method 2: $\int_C \nabla f \cdot d\mathbf{r} = f(1,2) - f(0,0) = 9.$

Proof of the fundamental theorem

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_C f_x\,dx + f_y\,dy = \int_{t_0}^{t_1} \left[f_x(x(t),y(t))\,\frac{dx}{dt} + f_y(x(t),y(t))\,\frac{dy}{dt} \right] dt \\ &= \int_{t_0}^{t_1} \frac{d}{dt} f(x(t),y(t))\,dt = f(x(t),y(t))|_{t_0}^{t_1} = f(P_1) - f(P_0) \quad \blacksquare\end{aligned}$$

The third equality above follows from the chain rule.

Significance of the fundamental theorem

For gradient fields $\mathbf{F}$ the work integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of the path.
We call such a line integral *path independent*.
The special case of this for closed curves C gives:

$$\mathbf{F} = \nabla f \;\Rightarrow\; \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{(proof below).}$$

Following physics, where a conservative force does no work around a closed loop, we say $\mathbf{F} = \nabla f$ is a *conservative* field.

Example 1: If $\mathbf{F}$ is the electric field of an electric charge it is conservative.

Example 2: The gravitational field of a mass is conservative.

Differentials: Here we can use differentials to rephrase what we've done before. First recall:

a) $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} \;\Rightarrow\; \nabla f \cdot d\mathbf{r} = f_x\,dx + f_y\,dy.$

b) $\displaystyle\int_C \nabla f \cdot d\mathbf{r} = f(P_1) - f(p_0).$

Using differentials we have $df = f_x\,dx + f_y\,dy$. (This is the same as $\nabla f \cdot d\mathbf{r}$.) We say $M\,dx + N\,dy$ is an *exact differential* if $M\,dx + N\,dy = df$ for some function f.

As in (b) above we have $\displaystyle\int_C M\,dx + N\,dy = \int_C df = f(P_1) - f(P_0).$

Proof that path independence is equivalent to conservative

We show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{is path independent for any curve } C$$

is equivalent to

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{for any closed path.}$$

This is not hard, it is really an exercise to demonstrate the logical structure of a proof showing equivalence. We have to show:

i) Path independence $\Rightarrow$ the line integral around any closed path is 0.

ii) The line integral around all closed paths is 0 $\Rightarrow$ path independence.

i) Assume path independence and consider the closed path C shown in figure (i) below. Since the starting point P_0 is the same as the endpoint P_1 we get $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = f(P_1) - f(P_0) = 0$ (this proves (i)).

ii) Assume $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve. If C_1 and C_2 are both paths between P_0 and P_1 (see fig. 2) then $C_1 - C_2$ is a closed path. So by hypothesis

$$\oint_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0 \;\Rightarrow\; \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

That is the line integral is path independent, which proves (ii).

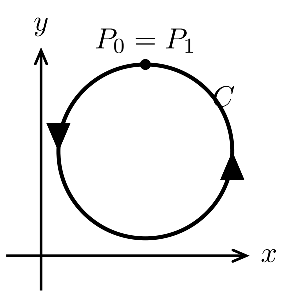


Figure (i)

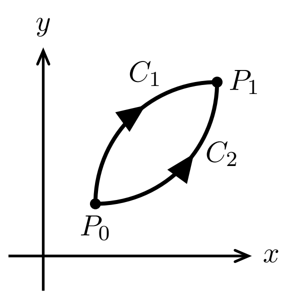


Figure (ii)