18.02 Problem Set 12, Part II Solutions

1. $\vec{F} = \langle \frac{-z}{x^2+z^2}, y, \frac{x}{x^2+z^2} \rangle = \langle P, Q, R \rangle$.

(a) $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-z}{x^2+z^2} & y & \frac{x}{x^2+z^2} \end{vmatrix} = \hat{i}(0-0) - \hat{j}(\frac{\partial}{\partial x}(\frac{x}{x^2+z^2}) + \frac{\partial}{\partial z}(\frac{z}{x^2+z^2})) + \hat{k}(0-0)$
$= \hat{j}(\frac{-1}{x^2+z^2} + \frac{2x^2}{(x^2+z^2)^2} - \frac{1}{x^2+z^2} + \frac{2z^2}{(x^2+z^2)^2}) = \frac{2}{x^2+z^2} - 2\frac{x^2+z^2}{(x^2+z^2)^2} = \mathbf{0}$ (for $x^2+z^2 > 0$).

(b) $\oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_1} P\,dx + Q\,dy$ (since $dz = 0$). C_1: $x = \cos t$, $y = \sin t$, $z = 1$, $0 \le t \le 2\pi$, so $dx = -\sin t\,dt$, $dy = \cos t\,dt$. $P(\cos t, \sin t, 1) = \frac{-1}{1+\cos^2 t}$, $Q(\cos t, \sin t, 1) = \sin t$. So $\oint_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} ((-\frac{1}{1+\cos^2 t})(-\sin t) + \sin t \cos t) dt$
$= -\tan^{-1}(\cos t)|_0^{2\pi} + \frac{1}{2}\sin^2 t|_0^{2\pi} = -(\tan^{-1}(1) - \tan^{-1}(1)) + \frac{1}{2}(0-0) = 0$.

(c) No, Stokes' Theorem does not apply to C_2, since any capping surface S for C_2 will have to have a point $(0, b, 0)$ on where the y-axis ($x = z = 0$) intersects S, and $\vec{\nabla} \times \vec{F}$ is not defined for $x = z = 0$.

(d) C_2: $x = \cos t$, $y = 0$, $z = \sin t$, for $0 \le t \le 2\pi$, so $dx = -\sin t\,dt$, $dy = 0$, $dz = \cos t\,dt$. Thus $\oint_{C_2} \vec{F} \cdot d\vec{r} = \oint_{C_2} P\,dx + R\,dz = \int_0^{2\pi} ((\frac{-\sin t}{1})(-\sin t) + (\frac{\cos t}{1})(\cos t)) dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = \int_0^{2\pi} 1\,dt = 2\pi \ne 0$

2 (a) $\vec{\nabla} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix} = \hat{i}(\frac{\partial}{\partial y}(\frac{z}{r^2}) - \frac{\partial}{\partial z}(\frac{y}{r^2}))) -$
$\hat{j}(\frac{\partial}{\partial x}(\frac{z}{r^2}) - \frac{\partial}{\partial z}(\frac{x}{r^2})) + \hat{k}((\frac{\partial}{\partial x}(\frac{y}{r^2}) - \frac{\partial}{\partial y}(\frac{x}{r^2})))$ where $r^2 = x^2 + y^2 + z^2$.
For the $\hat{i}$ component we get $(\frac{\partial}{\partial y}(\frac{z}{r^2}) - \frac{\partial}{\partial z}(\frac{y}{r^2}))) = -2r^{-3}(z\,2y - y\,2z) = 0$, and similarly for the other two components. Thus $\vec{\nabla} \times \vec{G} = \mathbf{0}$ (for $r > 0$).

(b) Yes. In this case we can take a capping surface S for C that avoids the origin, and then Stokes' Theorem applies to give $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\,dS = 0$.

(c) $\mathbb{R}^3 - \{y\text{-axis}\}$ is not simply connected, but $\mathbb{R}^3 - \{\mathbf{0}\}$ is simply connected.

3 (a) We need to show that $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{F} = 0$. $\rho = 1$, constant, so $\frac{\partial \rho}{\partial t} = 0$; and
$\nabla \cdot \mathbf{F} = (\frac{\partial}{\partial x})(z \sin t) + (\frac{\partial}{\partial y})(-z \cos t) + (\frac{\partial}{\partial z})(-x \sin t + y \cos t) = 0$.

(b) $\nabla \cdot \mathbf{F} = 0$ for all (x, y, z, t) implies that the flux though any closed surface S is zero, by the Divergence Theorem.

4 (a) $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z \sin t & -z \cos t & -x \sin t + y \cos t \end{vmatrix} =$

$\mathbf{i}\,(2 \cos t) - \mathbf{j}\,(-2 \sin t) + \mathbf{k}\,(0) = 2\,\langle \cos t, \sin t, 0 \rangle = 2\,\mathbf{n}_t$.

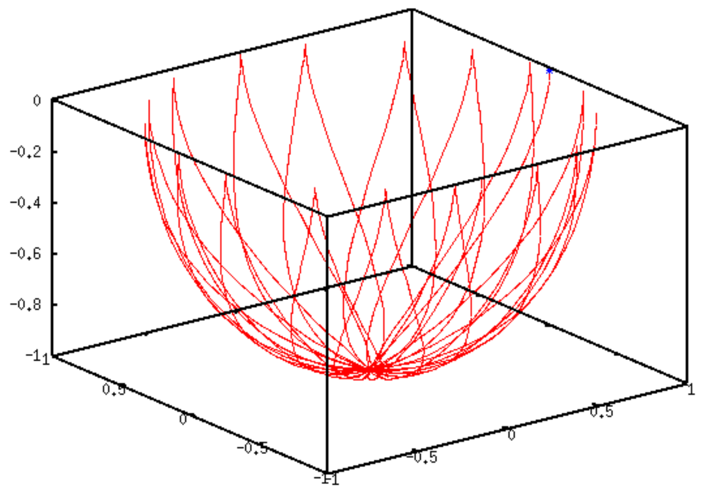
b) Direction of $\nabla \times \mathbf{F} = \langle \cos t, \sin t, 0 \rangle$, $|\,\nabla \times \mathbf{F}\,| = 2 = 2\,\omega_{\text{max}}$, so $\omega_{\text{max}} = 1$ in $\frac{\text{rad}}{\text{unit time}}$.

c) $\mathbf{n}_t \cdot \mathbf{v} = \langle \cos t, \sin t, 0 \rangle \cdot \langle z \sin t, -z \cos t, -x \sin t + y \cos t \rangle = z \sin t \cos t - z \cos t \sin t = 0$ for all t. At time t, the plane of fastest spin is $\mathcal{P}_t$.

d) Pure rotational flow in $\mathcal{P}_t$ (at 1 rad./unit time)

e) The fluid is rotating in the planes $\mathcal{P}_t$ while the planes $\mathcal{P}_t$ are rotating around the z-axis. The resulting flow will involve some kind of swirling pattern. Anyone riding the flow will probably regret it (unless unusually resistant to motion sickness).

The graph of a representative flow path – obtained from a DE numerical approximation program applied to the 3×3 ODE system $\mathbf{r}' = \mathbf{v}$ – is given below:



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