V15.1 Del Operator

1. Symbolic notation: the del operator

To have a compact notation, wide use is made of the symbolic operator “del” (some call it “nabla”):

$$\nabla \;=\; \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k} \tag{1}$$

Recall that the “product” of $\dfrac{\partial}{\partial x}$ and the function $M(x,y,z)$ is understood to be $\dfrac{\partial M}{\partial x}$. Then we have

$$\text{grad } f \;=\; \nabla f \;=\; \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} \tag{2}$$

The divergence is sort of a symbolic scalar product: if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$,

$$\text{div } \mathbf{F} \;=\; \nabla \cdot \mathbf{F} \;=\; \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \tag{3}$$

while the curl, as we have noted, as a symbolic cross-product:

$$\text{curl } \mathbf{F} \;=\; \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} .$$

Notice how this notation reminds you that $\nabla \cdot \mathbf{F}$ is a scalar function, while $\nabla \times \mathbf{F}$ is a vector function.

We may also speak of the Laplace operator (also called the “Laplacian”), defined by

$$\text{lap } f \;=\; \nabla^2 f \;=\; (\nabla \cdot \nabla)\, f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} . \tag{5}$$

Thus, Laplace’s equation may be written: $\nabla^2 f = 0$. (This is for example the equation satisfied by the potential function for an electrostatic field, in any region of space where there are no charges; or for a gravitational field, in a region of space where there are no masses.)

In this notation, the divergence theorem and Stokes’ theorem are respectively

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \;=\; \iiint_D \nabla \cdot \mathbf{F}\, dV \qquad\qquad \oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \tag{6}$$

Two important relations involving the symbolic operator are:

$$\text{curl (grad} f) \;=\; \mathbf{0} \qquad\qquad \text{div curl } \mathbf{F} \;=\; 0 \tag{7}$$

$$\nabla \times \nabla f \;=\; \mathbf{0} \qquad\qquad \nabla \cdot \nabla \times \mathbf{F} \;=\; 0 \tag{7$'$}$$

The first we have proved (it was part of the criterion for gradient fields); the second is an easy exercise. Note however how the symbolic notation suggests the answer, since we know that for any vector $\mathbf{A}$, we have

$$\mathbf{A} \times \mathbf{A} \;=\; \mathbf{0}, \qquad\qquad \mathbf{A} \cdot \mathbf{A} \times \mathbf{F} \;=\; 0 ,$$

and (7′) says this is true for the symbolic vector ∇ as well.