Testing for a Conservative Field

Let $\mathbf{F} = (3x^2y + az)\mathbf{i} + x^3\mathbf{j} + (3x + 3z^2)\mathbf{k}$.

1. For what value or values of a is $\mathbf{F}$ conservative?

Answer: We know $\mathbf{F}$ is conservative if it's continuously differentiable for all x, y, z and curl$F = 0$. We easily verify that $\mathbf{F}$ is continuously differentiable as required.

$$\text{curl}F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2y + az) & x^3 & (3x + 3z^2) \end{vmatrix} = 0\mathbf{i} - (3 - a)\mathbf{j} + (3x^2 - 3x^2)\mathbf{k} = (a - 3)\mathbf{j}.$$

If $a = 3$, curl$\mathbf{F} = 0$ and so $\mathbf{F}$ must be conservative.

Answer: $a = 3$.

2. Assuming a has the value(s) found in (1), find a potential function f for which $\mathbf{F} = \boldsymbol{\nabla} f$.

Answer: As usual, there are two ways to find such a potential function. For variety, we'll use the second method.

Assume that $\mathbf{F} = \boldsymbol{\nabla} f$. Then $f_x = 3x^2y + 3z$, so we have $f = x^3y + 3xz + g(y, z)$ for some function g.

Combine this with the fact that $f_y = x^3$ to get $x^3 + g_y = x^3$ so $g(y, z) = h(z)$ is constant with respect to y.

Finally, $f_z = 3x + h'(z) = 3x + 3z^2$ implies $h(z) = g(y, z) = z^3 + C$.

We conclude that $f(x, y, z) = x^3y + 3xz + z^3 + C$.

We can now calculate $f_x = 3x^2y + 3z$, $f_y = x^3$ and $f_z = 3x + 3z^2$ to check that $\mathbf{F} = \boldsymbol{\nabla} f$.

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