18.03 Class 17, March 12, 2010

Linearity and time invariance

[1] RLC
[2] Superposition III
[3] Time invariance
[4] Review of solution methods

[1] We've spent a lot of time with $mx'' + bx' + cx = q(t)$. There are many other systems modeled by this equation. For example here is a series RLC circuit:

```
           ____
          /    \                                            ||
   ------(      )-----/\/\/\/-------------------------------||------
  |       \____/                                            ||      |
  |                                                                 |
  |                                                                 |
  |_________________________________________________________________|
```

I = current; the same everywhere
V = voltage increase across the power source
V_R = voltage drop across the resistor
V_C = voltage drop across the capacitor

Input signal: V
System response: I

KVL: $V_R + V_C = V$

Component behavior:

$V_R = R I$ $\qquad$ $V_C' = (1/C) I$

Differentiate KVL : $\qquad$ $V_R' + V_C' = V'$

so $\qquad$ $R I' + (1/C) I = V'$

In Lecture 3 we offered this as an example of a first order LTI system.

Now let's add another component, an inductor.

```
           ____
          /    \                            __ ___ __           ||
   ------(      )-----/\/\/\/--------/  X   X  \-----------||------
  |       \____/                        V   V              ||      |
  |                                                                 |
  |                                                                 |
  |_________________________________________________________________|
```

The voltage drop across an inductor depends not on the current but rather on the derivative of the current:

$$V_L = L I' \qquad \text{so} \quad V_L' = L I'' .$$

KVL now says $\qquad V_L + V_R + V_C = V$

so $\qquad L I'' + R I' + (1/C) I = V' \qquad (*)$

The system serves as an OPERATOR:

$$L D^2 + R D + (1/C) \, Id$$

takes the current I , as a function of time, and gives the derivative of the impressed voltage. (I have to use a different symbol for the identity operator here, so it doesn't get confused with current. I chose "Id.")

[2] Suppose you want to solve $u'' - 4u = \cosh(2t)$

Remember, I can write the left hand side as $p(D) u$ where $p(s) = s^2 - 4$.

First you have to know what $\cosh(t)$ is :

$$\cosh(x) = (e^x + e^{-x}) / 2$$
$$\sinh(x) = (e^x - e^{-x}) / 2$$

The right hand side is a linear combination of exponentials, each of which we know very well how to deal with.

Superposition III: $p(D) (c_1 x_1 + c_2 x_2) = c_1 p(D) x_1 + c_2 p(D) x_2$
[Slide]

Therefore, if $p(D) x_1 = q_1(t)$ and $p(D) x_2 = q_2(t)$, then

$x = c_1 x_1 + c_2 x_2$ solves $p(D) x = c_1 q_1(t) + c_2 q_2(t)$

Proof of Super III: [Slide]

$$p(D) x = a_n x^{(n)} + \dots + a_1 x' + a_0 x$$

$$\begin{array}{ll} a_0 \,] & x = c_1 x_1 + c_2 x_2 \\ a_1 \,] & x' = c_1 x_1' + c_2 x_2' \\ a_2 \,] & x'' = c_1 x_1'' + c_2 x_2'' \\ & \quad \dots \\ \hline & p(D) x = c_1 p(D) x_1 + c_2 p(D) x_2 \end{array}$$

So we should solve separately: $u_1'' - 4u_1 = e^{2t}$, $u_2'' - 4u_2 = e^{-2t}$

Try to apply ERF: $p(s) = s^2 - 4$, $p(2) = 0$ and $p(-2) = 0$:
so we must use ERF/R : $p'(s) = 2s$, $p'(2) = 4$, $p'(-2) = -4$,

$u_1 = te^{2t}/4$ solves $u_1'' - 4u_1 = e^{2t}$
$u_2 = te^{-2t}/(-4)$ solves $u_2'' - 4u_2 = e^{-2t}$

so $u = (t/4) (e^{2t} - e^{-2t}) / 2 = (t/4) \sinh(2t)$ is a solution.

Super III contains the earlier episodes.
With $q_2 = 0$, this gives us "Superposition II,"
With $q_1 = 0$ and $q_2 = 0$, we get" Superposition I."

The property Superposition III says that $p(D)$ is a LINEAR OPERATOR.

[3] You may be worried that we always say that we are interested in sinusoidal input, but then we always consider $A \cos(\omega t)$, with no phase lag. Isn't that kind of restrictive?

Ans: No. Here's why.

First, if the right hand side is a sine, there is a special thing you can do:

Example: $x'' + 9x = \sin(2t)$

This is the IMAGINARY part of $z'' + 9z = e^{2t}$: $p(s) = s^2 + 9$,
$p(2i) = 9 - 4$, $z_p = e^{2it} / (9-4)$, so
$x_p = \mathrm{Im} (e^{2it}/(9-4)) = (1/(9-4) \sin(2t)$

This is worth remembering, by the way: if you drive a harmonic oscillator, there is no phase lag:

$x'' + \omega_n^2 x = \cos(\omega t)$: $x_p = \cos(\omega t) / (\omega_n^2 - \omega^2)$

$x'' + \omega_n^2 x = \sin(\omega t)$: $x_p = \sin(\omega t) / (\omega_n^2 - \omega^2)$

But there is a better and more general reason:
We are studying the response of a system which
is not changing: the coefficients are constant. So if I start the signal a little bit later, all that happens is that the system response is correspondingly delayed.

If we shift the graph of a function $x(t)$ to the right by a units, we get the graph of the function $x(t-a)$.

TIME INVARIANCE: If $p(D) y(t) = q(t)$, then $p(D) y(t-a) = q(t-a)$
[Slide]

To solve $p(D) x = q(t-a)$, first solve $p(D) y = q(t)$. Then $x(t) = y(t-a)$.

It's very important that the coefficients are *constant* for this to work.

Example: $x'' + x' + 6x = \cos(2t - \pi/3)$

Method: (1) Rewrite the right hand side as f(t-a):

cos(2t - pi/3) = cos(2(t - pi/6)) so f(t) = cos(2t) and a = pi/6 .

(2) Solve y" + y' + 6y = f(t) = cos(2t) ;

z" + z' + 6z = e^{2it} ,

p(s) = s^2 + s + 6 , p(2i) = -4 + 2i + 6 = 2 + 2i = 2sqrt(2) e^{i pi/4},

z_p = e^{2it} / 2(1+i) = (1/2sqrt(2)) e^{- i pi/4} e^{2it}
    = (1/2sqrt 2) e^{i(2t - pi/4)} ,

y_p = Re(z_p) = (1/2sqrt 2) cos(2t - pi/4)

(3) Then a solution to the original equation is

x(t) = y(t-a) = y_p(t - pi/6) = (1/2sqrt 2) cos(2(t - pi/6) - pi/4) .
                              = (1/2sqrt 2) cos((2t - pi/3) - pi/4) .

Be careful here: the (t - pi/6) goes in place of t , so gets multiplied by the circular frequency.

In any case: [Slide] The amplitude and the phase lag of the solution to

p(D) x = A cos(omega t - phi_0)

depend only on omega and A , and not on phi_0 .

If there is a system you are studying, it has a gain and a phase lag, functions of the circular frequency of the input signal. These are the same for any sinusoidal input with the given frequency. They don't depend on amplitude or phase of input.

[4] Brief summary of solution methods

We are looking at LTI operators. Solutions x_h of p(D) x = 0 are linear combinations of e^{rt} where r is a root of the characteristic polynomial: "modes." If r is a double root you have to add te^{rt} (and so on). If r = a+bi is a non-real root (and p(s) has real coefficients) the r-conjugate is also a root, and these two conjugate modes combine to give e^{at} cos(bt) and e^{at} sin(bt) .

If all roots have negative real part, then all these solutions decay to zero as t ---> infinity : they are transients.

To solve p(D) x = q(t) , find some solution x_p ; then the general solution is x_p + x_h : all solutions converge as t ---> infinity.

q(t) = e^{rt} : ERF or ERF/R

q(t) = cos : complex replacement reduces to exponential input signal.

q(t) = exp x cos : ditto.

q(t) = sinusoidal : use time invariance to shift to cosine.

q(t) = polynomial : undetermined coefficients, preceded by reduction of order if necessary

q(t) = (anything) x exponential : variation of parameters leads to a new differential equation in which the exponential has been eliminated from the right hand side.

q(t) = linear combination: Superposition III .