18.03 Class 28, Apr 14, 2010

Laplace Transform III: Second order equations; completing the square.

1. Question period: f', FTC
2. L[f'_r(t)]
3. s-derivative rule
4. Second order equations

Question period:

[1] There were good questions from the class about finding the generalized derivative of

f(t) = u(t) cos(omega t) .

Let's do this again. The graph shows that f(t) is continuous except at t = 0 . The regular part of the derivative is

f'_r(t) = - omega u(t) sin(omega t)

and the singular part is

f'_s(t) = delta(t)

You can also apply the product rule:

f'(t) = - omega u(t) sin(omega t) + delta(t) cos(omega t)

= - omega u(t) sin(omega t) + delta(t)

since cos(omega 0) = 1 .

In order for the "fundamental theorem of calculus"

integral_a^c f'(t) dt = f(c) - f(a)

to be true, you MUST use the generalized derivative.

For example, take a = -1, b = 1, f(t) = u(t) .
If you say u'(t) = 0 , then

integral_a^c u'(t) dt = 0 , not 1

... while

integral_a^c delta(t) dt = 1

[2] At the end of class on Monday I took a regular function f(t) (with f(0) = 0 for t < 0) whose only jump in value is at t = 0 , and found L[f'_r(t)] :

f'(t) = f'_r(t) + f'_s(t)

$= f'_r(t) + (f(0+) - f(0-))\ \delta(t)$

$= f'_r(t) + f(0+)\ \delta(t)$

So:

$$s F(s) = L[f'(t)] = L[f'_r(t)] + f(0+)$$

or

$$L[f'_r(t)] = s F(s) - f(0+)$$

Example: Solve $x' + 2x = 4t$ with initial condition $x(0) = 1$ using Laplace transform.

Some interpretation is needed: Since we are always assuming $x(0) = 0$ for $t < 0$, we must mean $x'(0+) = 0$.

So there is a jump at $t = 0$ in the value of $x(t)$. This must produce a delta function in the derivative. Where is it in the equation? Missing! There's no solution to this equation if x' is interpreted as the generalized derivative. What is intended is the ordinary derivative:

$$x'_r + 2x = 4t \text{ with initial condition } x(0+) = 1 .$$

Apply LT:

$$sX - 1 + 2X = 4/s^2$$

$$X = 4/s^2(s+2) + 1/(s+2)$$

Work on these separately. The first is $L[e^{-2t}]$.

For the second we need a fancier version of partial fractions:

$$4/s^2(s+2) = (as + b)/s^2 + c/(s+2)$$

$$= a/s + b/s^2 + c/(s+2)$$

Find b and c by coverup:

$4/(s+2) = b + s (...)$, so with $s = 0$ you find $b = 2$.

$4/s^2 = (s+2) (...) + c$, so with $s = -2$ you find $c = 1$

To find a you have to do something different. How about setting s = anything else, say 1:

$4/3 = a + 2 + 1/3$ or $a = -1$

$$1/s^2(s+2) = -1/s + 2/s^2 + 1/(s+2)$$

$$x = - 1 + 2t + 2e^{-2t}$$

We could have gotten this more easily using undetermined coefficients!

[3] Another rule: The s-derivative rule :

$$F'(s) = (d/ds) \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty (-t e^{-st}) f(t)\, dt$$

which is the Laplace transform of $- t f(t)$. Thus:

$$L[t f(t)] = - F'(s)$$

Compare this with the t-derivative rule $L[f'(t)] = s F(s)$.

Example: $f(t) = \sin(\omega t)$, $F(s) = \omega / (s^2 + \omega^2)$.

$$L[t \sin(t)] = - F'(s) = 2 \omega s / (s^2 + \omega^2)^2$$

[4] Second order differential equations.

$L[f''(t)] = ?$ Let $g(t) = f'(t)$: $G(s) = sF(s)$

$L[f''(t)] = L[g'(t)] = sG(s) = s^2 F(s)$

We will be careful to use this only when $f(0+) = 0$, so that $f'(t)$ doesn't have a delta function in it; we are not going to differentiate the delta function in this course.

Example: Find the unit impulse response for $D^2 + 2D + 2I$

$$w'' + 2w' + 2w = \delta(t) \text{ , rest initial conditions}$$

$$s^2W + 2sW + 2W = 1$$

$$W = 1/(s^2 + 2s + 2)$$

Technique: complete the square:

$$W = 1/((s+1)^2 + 1)$$

$$\sin(t) \longrightarrow 1/(s^2 + 1)$$

so by the s-shift rule

$$e^{-t} \sin(t) \longrightarrow W(s)$$

and $w(t) = u(t) e^{-t} \sin(t)$

TERMINOLOGY: unit impulse response = weight function
L[unit impulse response] = transfer function

GENERAL FACT: The transfer function of $p(D)$ is $W(s) = 1/p(s)$:

$$a_n w^{(n)} + \dots + a_0 w = \delta(t)$$

$$a_n s^n W + \dots + a_0 W = 1$$

$$p(s) W = 1$$

so $$LT[w] = 1/p(s)$$

or $$LT^{-1}[1/p(s)] = w(t)$$