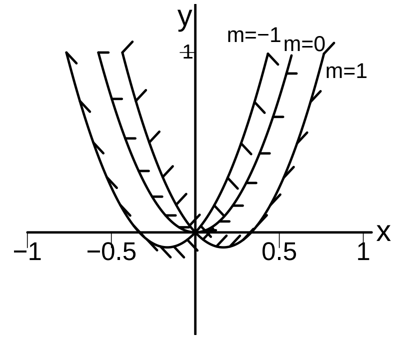
Practice Final Exam



Direction field

1. *For the DE* $\dfrac{dy}{dx} = -\dfrac{y}{x} + 3\,x$:
a) Sketch the direction field for this DE, using (light or dotted) isoclines for the slopes -1 and 0.

See picture at right
Isoclines: $y' = -\frac{y}{x} + 3x = m \Rightarrow y = 3x^2 - mx$.
(Note problem at $(0, 0)$.)

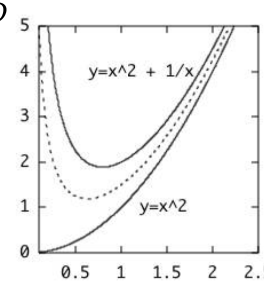
b) *For the solution curve passing through the point (1,2): if Euler's method with step-size* $h = 0.1$ *was used to approximate* $y(1.1)$, *would the approximation come out too high or too low? Explain.*

If $y(1) = 2$ then $y'(1) = -\frac{2}{1} + 3 \cdot 1 = 1$.
$y'' = -\frac{xy' - y}{x^2} + 3 \Rightarrow y''(1) = 4 > 0 \Rightarrow$ concave up
$\Rightarrow$ estimate is too low.

c) *Compute the Euler approximation to* $y(1.1)$ *using step-size* $h = 0.1$.

Euler: $y_1 = y_0 + hf(x_0, y_0) = 2 + 0.1 \cdot f(1, 2) = 2.1$. $y(1.1) \approx 2.1$

d) *The functions* $y_1 = x^2$ *and* $y_2 = x^2 + \frac{1}{x}$ *are solutions to this DE. If* $y = y(x)$ *is the solution satisfying the IC* $y(1) = 1.5$, *show that* $100 \leq y(10) \leq 100.1$. *D include the equal signs in this inequality? Why or why not?*



The picture at right shows the plots of the 2 given solutions.
The dotted line indicates that (by the E&U) the solution with IC (1,1.5) must stay between these two solutions.
$\Rightarrow y_1(10) < y(10) < y_2(10) \Rightarrow 100 < y(10) < 100.1$.
As stated, E&U (= Existence & Uniqueness theorem) shows we don't need the equal signs.

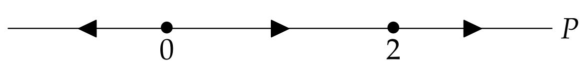
e) *Find the general solution the DE and verify the prediction of part (b).*

The easiest way to find the general solution is to use the 2 solutions given in part (d). The equation is first order linear, so the general solution to the homogeneous equation is $c(y_2 - y_1) = c \cdot \frac{1}{x}$. Thus the general solution is $\boxed{y = x^2 + c \cdot \frac{1}{x}.}$

2. *Suppose that a population of variable size (in some suitable units)* $p(t)$ *follows the growth law* $\dfrac{dp}{dt} = p^3 - 4p^2 + 4p$. *Without solving the DE explicitly:*

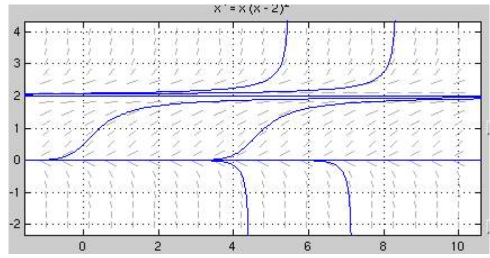
a) *Find all critical points and classify each according to its stability type using a phase-line diagram.*

$p^3 - 4p^2 + 4p = p(p-2)^2 \Rightarrow$ critical points are $p = 0$ and $p = 2$.
By looking at the phase line we see that 0 is unstable and 2 is semi-stable.
Here is the phase line.



b) *) Draw a rough sketch (on p-vs.-t axes) of the family of solutions. What happens to the population in the long-run if it starts out at size 1 unit; at size 3 units ?*

By hand a rough sketch is fairly easy: First draw horizontal lines at $p = 0$ and $p = 2$ (the equilibrium solutions). Next draw the solutions above $p = 2$ as curves curving upward, those between 0 and 2 are 'logistic-like' going from 0 to 2 and those below 0 curve down. Below is a dfield plot of the system. In the long run if $p(0) = 1$ then $p \to 2$ and if $p(0) = 3$ then $p \to \infty$.



c) *Explain why the rate information given by the DE was all we needed to get the answer to part (b).*

The rate information was all we needed to draw the phase line and determine the critical points and their stability types.

3. *Let* $p(D) = D^2 + b\,D + 5$ *where* $D = \frac{d}{dt}$. *a) For what range of the values of* b *will the solutions to* $p(D)y = 0$ *exhibit oscillatory behavior?*

Characteristic polynomial: $p(s) = s^2 + bs + 5$. Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 20}}{2}$.

Solutions are oscillatory if r is complex. i.e. if $b^2 - 20 < 0 \Rightarrow b < \sqrt{20}$.

b) *For* $b = 4$, *solve the DE's (i)* $p(D)y = 4\,e^{2t}\sin t$
(ii) $p(D)y = 4\,e^{2t}\cos t$
using the Exponential Response formula. Write your answers in both amplitude-phase and rectangular form.

Complex replacement: $e^{2t}\sin t = \text{Im}(e^{(2+i)t}) \Rightarrow$ solve $p(D)z = e^{(2+i)t}$, where for (i) $y = \text{Im}\,z$, and for (ii) $y = \text{Re}\,z$.

$p(2+i) = (2+i)^2 + 4(2+i) + 5 = 16 + 8i = 8(2+i)$

$\Rightarrow$ (ERF) $z_p = 4\,\dfrac{e^{(2+i)t}}{8(2+i)}$.

Amplitude-phase: $z_p = \dfrac{1}{2\sqrt{5}} e^{2t} e^{i(t-\phi)}$, where $\phi = \tan^{-1}(1/2)$ (first quadrant)

$$\Rightarrow \boxed{\text{(i) } y_p = \text{Im}(z_p) = \frac{1}{2\sqrt{5}} e^{2t}\sin(t-\phi), \quad \text{(ii) } y_p = \text{Re}(z_p) = \frac{1}{2\sqrt{5}} e^{2t}\cos(t-\phi)}$$

Rectangular: $z_p = \dfrac{1}{2(2+i)} e^{2t}(\cos t + i\sin t)\,\dfrac{2-i}{2-i}$

$$\Rightarrow \boxed{\text{(i) } y_p = \frac{1}{10} e^{2t}(2\sin t - \cos t)., \quad \text{(ii) } y_p = \frac{1}{10} e^{2t}(2\cos t + \sin t).}$$

c) *Given* $b = 2$, *for what* ω *does* $p(D)y = \cos\omega t$ *have the biggest response?*

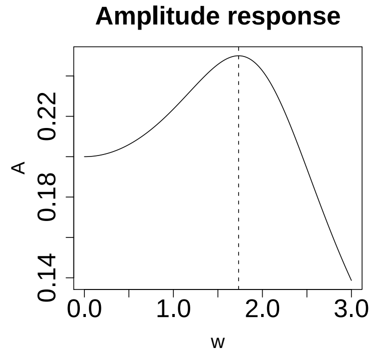
c) Complex replacement gives $p(D)z = e^{i\omega t}$, $y = \mathrm{Re}(z)$.

$\Rightarrow z_p = \dfrac{e^{i\omega t}}{5 - \omega^2 + 2\omega i} = A(w)e^{i(wt-\phi)}$,

where amplitude $A(\omega) = \dfrac{1}{\sqrt{(5-\omega^2)^2 + 4\omega^2}}$.

$A(w)$ has a max. when $(5-\omega^2)^2 + 4\omega^2$ has a min.

$\Rightarrow \boxed{\omega = \sqrt{3}}$ (practical resonance).



4. *Find the general solution to the DE* $(D^3 - 1)\, y = e^x$
Express the answer using real - valued functions only. Characteristic polynomial: $p(s) = s^3 - 1$.

Roots: r = cube roots of unity = 1, $e^{\pm 2\pi i/3} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$.

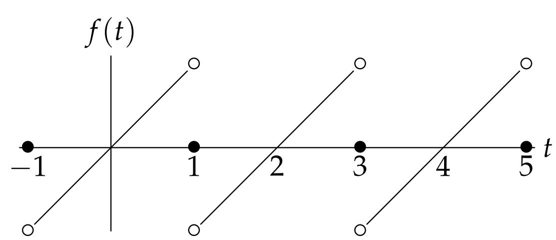
General real homogeneous solution: $\boxed{y_c = c_1 e^x + e^{-\frac{x}{2}}(c_2 \cos(\frac{\sqrt{3}}{2}x) + c_3 \sin(\frac{\sqrt{3}}{2}x)).}$

Particular solution: $p(1) = 0$, $p'(1) = 3 \neq 0 \Rightarrow \boxed{y_p = \dfrac{xe^x}{p'(1)} = \dfrac{1}{3}xe^x}$

(Resonant Response formula) General solution: $\boxed{y = y_p + y_c.}$

5. *For* $f(t) = t$ *on* $-1 < t < 1$, *periodic with period* $P = 2$:
a) *Sketch* f *over three or more full periods* P
Choose endpoint values that show where the Fourier series expansion will converge (without computing the Fourier series).

The Fourier series will converge to f except at the points of discontinuity, where it will converge to the midpoint of the jump.



b) *Compute the Fourier series of* f

Since $f(t)$ is odd, we have sine terms only. Then
$b_n = 2\int_0^1 t \sin(n\pi t)\, dt = 2\left(-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{n^2\pi^2} \Big|_0^1 \right) = -2\frac{(-1)^n}{n\pi}$

$$\Rightarrow \boxed{f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi t).}$$

c) *Compute the steady-periodic solution to the DE* $x''(t) + 10\, x(t) = f(t)$. *Does near-resonance occur in this situation? If so, which frequency in the 'driving force'* $\tilde{f}_{\text{odd}}$ *pro-*

duces it?

Using complex replacement and the Exponential Response formula, the steady-periodic solution to $x'' + 10x = \sin(at)$ is $x = \dfrac{\sin(at)}{10 - a^2}$ (provided $10 - a^2 \neq 0$).
Superposition (using the infinite Fourier series) gives that the steady-periodic solution to the DE ($x'' + 10x = \tilde{f}_{\text{odd}}$) is $\quad x_{\text{sp}} = \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n(10 - n^2\pi^2)} \sin(n\pi t)$.
Since $\pi^2 \approx 9.87 \approx 10$ the $n = 1$ term produces the largest response (i.e. 'near resonance').

6. a) *Suppose that starting at $t = 0$ a radioactive material is continuously flowing into a container at a rate $f(t)$ in curies per unit time, and that one uses the standard exponential model for continuous radioactive decay, with rate constant k (in $\frac{1}{\text{time}}$). Let $R = R(t)$ denote the total amount of radioactive material in the container at time t. Give the DE for $R(t)$ and solve it (in terms of k, $f(t)$ and $R_0 = R(0)$).*

DE: $R' + kR = f(t),\ R(0) = R_0$.
Solution: (first order linear)
$R(t) = e^{-kt}\left(\int_0^t f(u)e^{ku}\,du + R_0\right)$.

b) *Show that the solution satisfying the IC $R(0) = 0$ can be written as a convolution integral. What is the weight function w in this case? What DE does w satisfy?*

If $R_0 = 0$ then $R(t) = e^{-kt}\int_0^t f(u)e^{ku}\,du = \int_0^t f(u)e^{-k(t-u)}\,du$ = the convolution $f * w$, where $w(t) = e^{-kt}$ is the weight function.
DE for w: $w' + kw = \delta$; with rest IC.

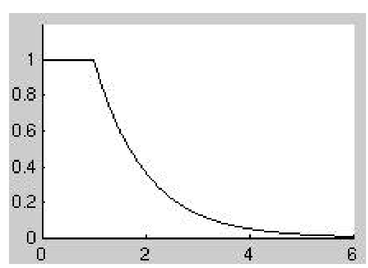
7. *Let $\mathcal{L}$ denote the Laplace transform (as usual). Derive the formula for $\mathcal{L}(\cos(t))$, by expressing $\cos(t)$ in terms of the complex exponential.*

We know $\mathcal{L}$ is linear and $\mathcal{L}(e^{at}) = \frac{1}{s-a}$.
$\cos t = \frac{1}{2}(e^{it} + e^{-it}) \Rightarrow \mathcal{L}(\cos t) = \frac{1}{2}(\mathcal{L}(e^{it}) + \mathcal{L}(e^{-it})) = \frac{1}{2}(\frac{1}{s-i} + \frac{1}{s+i}) = \frac{s}{s^2+1}$.

8. *Compute $f(t) = \mathcal{L}^{-1}\left(\dfrac{1}{s} - \dfrac{e^{-s}}{s(s+1)}\right)$ in both "u -form" and in "cases" form, and sketch the graph of f.*

Partial fractions: $\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$.

$$\begin{aligned}\mathcal{L}^{-1}\left(\tfrac{1}{s} - \tfrac{e^{-s}}{s(s+1)}\right) &= \mathcal{L}^{-1}\left(\tfrac{1}{s}\right) - \mathcal{L}^{-1}\left(\tfrac{e^{-s}}{s}\right) + \mathcal{L}^{-1}\left(\tfrac{e^{-s}}{s+1}\right) \\ &= 1 - u(t-1) + u(t-1)e^{-(t-1)} \\ &= \begin{cases} 1 & \text{if } 0 < t \le 1 \\ e^{-(t-1)} & \text{if } t \ge 1 \end{cases}\end{aligned}$$



9. *Use the Laplace transform method to solve the IVP's*

a) $y'' - y' - 2y = 0,\ y(0) = 0,\ y'(0) = 2$

Laplace transform: $(s^2 - s - 2)Y(s) - 2 = 0$
$\Rightarrow Y(s) = \frac{2}{s^2-s-2} = \frac{2}{(s-2)(s+1)}$.
Partial fractions: $Y(s) = \frac{2}{3}(\frac{1}{s-2} - \frac{1}{s+1})$.

Inverse Laplace: $y(t) = \frac{2}{3}(e^{2t} - e^{-t})$.

b) $y'' + 4y = \cos t, \;\; y(0) = y'(0) = 0$

Laplace transform: $(s^2+4)Y(s) = \frac{s}{s^2+1} \;\Rightarrow\; Y(s) = \frac{s}{(s^2+1)(s^2+4)}$.

Partial fractions: $Y(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \;\Rightarrow\; A = \frac{1}{3},\; B = 0,\; C = -\frac{1}{3},\; D = 0$.

Inverse Laplace: $y(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t$.

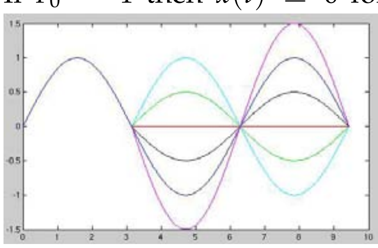
10. *Consider an undamped spring-mass system $Lx = x'' + x = f(t)$, where $f(t)$ is an external applied force, and suppose that the system starts out at time $t = 0$ at its equilibrium position $x = 0$ with a velocity $x'(0) = 1$ (in some suitable units). Using the Laplace transform method, solve for the position function $x = x(t)$ for the following forcing function $f(t)$:*
$f(t)$ is an impulsive *force of magnitude F_0 at time $t = \pi$ and $f(t) = 0$ otherwise.*
Graph the general solution. What happens in the special case F_0 = 1, and why?
Express your answer in u-form, and in 'cases' form as well.

Expressed in terms of δ, $f(t) = F_0\,\delta(t-\pi) \;\Rightarrow\; F(s) = F_0\,e^{-\pi s}$.

Laplace transform: $(s^2+1)X(s) - 1 = F_0\,e^{-\pi s} \;\Rightarrow\; X(s) = \frac{1}{s^2+1} + F_0\,\frac{e^{-\pi s}}{s^2+1}$.

Inverse Laplace:
$$\begin{aligned} x(t) &= \sin t + F_0\; u(t-\pi)\;\sin(t-\pi) \\ &= (1 - F_0\,u(t-\pi))\;\sin t = \begin{cases} \sin t & 0 \le t < \pi \\ (1-F_0)\;\sin t & \pi \le t \end{cases} \end{aligned}$$

If $F_0 = 1$ then $x(t) \equiv 0$ for $t \ge \pi$, i.e. the mass gets 'hammered to rest' at $t = \pi$.



magenta:F0=-.5
blue:F0=0
black:F0=.5
red:F0=1
green:F0=1.5
cyan:F0=2

11. *Find the general real solution to the DE system* $x' = x - 2y \qquad y' = 4x + 3y$ *using the eigenvalue/eigenvector method.*

Matrix equation: $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}\mathbf{x}$.

Characteristic equation: $p(\lambda) = \lambda^2 - 4\lambda + 11 = 0$.
Eigenvalues: $\lambda = 2 \pm \sqrt{7}i$.

Eigenvector (for $\lambda = 2 + \sqrt{7}i$): $\begin{pmatrix} -2 \\ 1+\sqrt{7}i \end{pmatrix}$

A complex solution $\tilde{\mathbf{x}} = e^{(2+\sqrt{7}i)t}\begin{pmatrix} -2 \\ 1+\sqrt{7}i \end{pmatrix}$

$$= e^{2t}(\cos\sqrt{7}t + i\sin\sqrt{7}t)\begin{pmatrix} -2 \\ 1+i\sqrt{7} \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} -2\cos\sqrt{7}t - 2i\sin\sqrt{7}t \\ \cos\sqrt{7}t - \sqrt{7}\sin\sqrt{7}t + i(\sin\sqrt{7}t + \sqrt{7}\cos\sqrt{7}t) \end{pmatrix}$$

Real and imaginary parts: $\mathbf{x_1} = \text{Re}(\mathbf{tx}) = e^{2t}\begin{pmatrix} -2\cos\sqrt{7}t \\ \cos\sqrt{7}t - \sqrt{7}\sin\sqrt{7}t \end{pmatrix}$

$\mathbf{x_2} = \text{Im}(\mathbf{x}) = e^{2t}\begin{pmatrix} -2\sin\sqrt{7}t \\ \sin\sqrt{7}t + \sqrt{7}\cos\sqrt{7}t \end{pmatrix}$

General solution: $v(x) = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.

12. *For the DE system* $\mathbf{x}' = A_a\mathbf{x}$ *with* $A_a = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$*:*

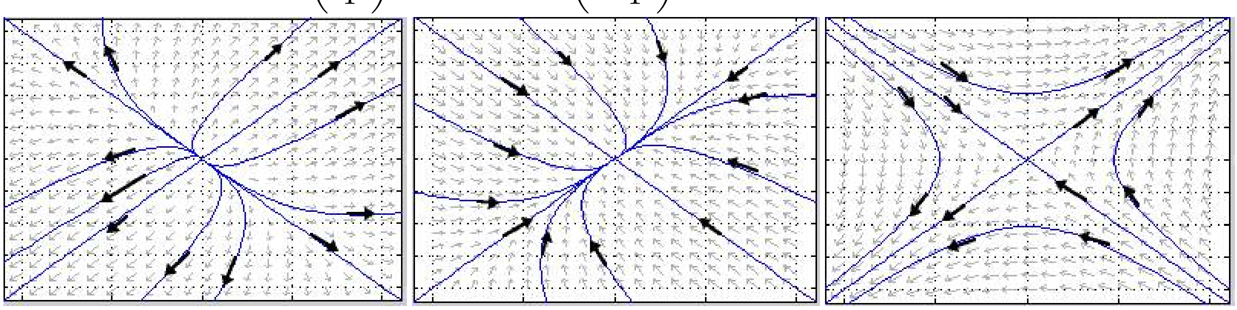
a) *Find the range of the values of a for which the critical point at (0,0) will be:*
(i) a source node (ii) a sink node (iii) a saddle.

Characteristic equation: $p(\lambda) = \lambda^2 - 2a\lambda + a^2 - 1 = 0$.
Eigenvalues: $\lambda = a \pm 1$.
(i) $a > 1 \Rightarrow \lambda_1, \lambda_2 > 0 \Rightarrow$ source node.
(ii) $a < -1 \Rightarrow \lambda_1, \lambda_2 < 0 \Rightarrow$ sink node.
(iii) $-1 < a < 1 \Rightarrow \lambda_1 > 0\, \lambda_2 < 0 \Rightarrow$ saddle.

b) *Choose a convenient value for a for each of the types above, solve, and sketch the trajectories in the vicinity of the critical point, showing the direction of increasing t.*

Sketches appear below. We choose $a = 2,\ 0,\ -2$.
(i) $a = 2$
Eigenvalues: $\lambda_1 = 3$ and $\lambda_2 = 1$.
Eigenvectors $\mathbf{v_1} = (1,1)^{\mathrm{T}}$ and $\mathbf{v_2} = (1,-1)^{\mathrm{T}}$.
Normal modes: $\mathbf{x_1} = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{x_2} = e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(ii) $a = -2$
Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -3$.
Eigenvectors $\mathbf{v_1} = (1,1)^{\mathrm{T}}$ and $\mathbf{v_2} = (1,-1)^{\mathrm{T}}$.
Normal modes: $\mathbf{x_1} = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{x_2} = e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
(iii) $a = 0$
Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = -1$.
Eigenvectors $\mathbf{v_1} = (1,1)^{\mathrm{T}}$ and $\mathbf{v_2} = (1,-1)^{\mathrm{T}}$.
Normal modes: $\mathbf{x_1} = e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{x_2} = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$



13. *For the DE system* $x' = x - 2y \qquad y' = 4x - x^3$*:*

a) *Compute the critical points of this system.*

Critical points: $y' = 0 \Rightarrow x(4-x^2) = 0 \Rightarrow x = 0, \pm 2$.
$\Rightarrow$ critical points are $(0,0)$, $(2,1)$, $(-2,-1)$.

b) *Find the type of each of the critical points using the linearized system which approximates this non-linear system and classify them according to their stability type and also their structural stability type. (Use the Jacobian.)*

Jacobian: $J(x,y) = \begin{pmatrix} 1 & -2 \\ 4-3x^2 & 0 \end{pmatrix}$.

Crit. pt. $(0,0)$: $J(0,0) = \begin{pmatrix} 1 & -2 \\ 4 & 0 \end{pmatrix}$

Characteristic equation: $p(\lambda) = \lambda^2 - \lambda + 8 = 0$.
Eigenvalues: $\frac{1}{2}(1 \pm \sqrt{-31}) \Rightarrow$ spiral source.
Unstable equilibrium, structurally stable.
Direction: test at $\mathbf{x} = (1,0)^T \Rightarrow \mathbf{x}' = (1,3)^T \Rightarrow$ counterclockwise.

Crit. pt. $(2,1)$: $J(2,1) = \begin{pmatrix} 1 & -2 \\ -8 & 0 \end{pmatrix}$

Characteristic equation: $p(\lambda) = \lambda^2 - \lambda - 16 = 0$.
Eigenvalues: $\frac{1}{2}(1 \pm \sqrt{65}) \Rightarrow$ saddle.
Unstable equilibrium, structurally stable.

For part (c) we'll want the eigenvectors:

$$\begin{pmatrix} 1-\lambda & -2 \\ -8 & -\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{rcrcl} (1-\lambda)a_1 & - & 2a_2 & = & 0 \\ -8a_1 & - & \lambda a_2 & = & 0 \end{array}$$

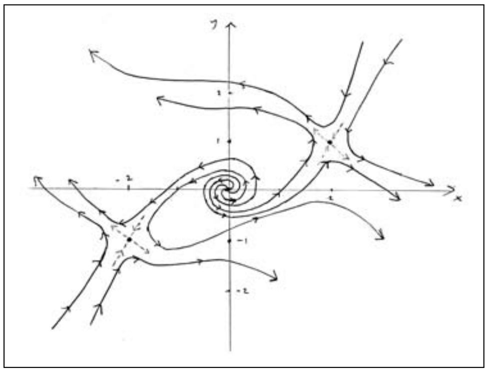
For $\lambda_1 = \frac{1+\sqrt{65}}{2} \Rightarrow \mathbf{v_1} = \begin{pmatrix} 4 \\ 1-\sqrt{65} \end{pmatrix} \approx \begin{pmatrix} 4 \\ -7.1 \end{pmatrix}$.

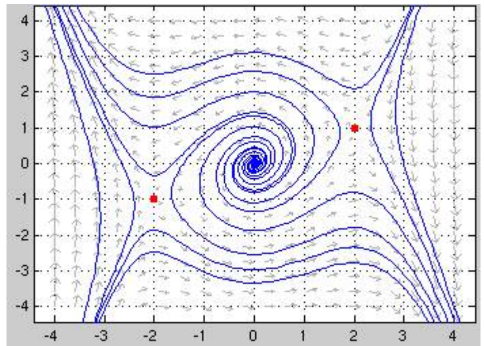
For $\lambda_2 = \frac{1-\sqrt{65}}{2} \Rightarrow \mathbf{v_2} = \begin{pmatrix} 4 \\ 1+\sqrt{65} \end{pmatrix} \approx \begin{pmatrix} 4 \\ 9.1 \end{pmatrix}$.

Crit. pt. $(-2,-1)$: $J(-2,-1) = \begin{pmatrix} 1 & -2 \\ -8 & 0 \end{pmatrix}$, same as $(2,1)$:
saddle, unstable equilibrium, structurally stable.

c) *Using the results of part(b), compute the eigenvectors as needed. Now put it all together into a reasonable sketch of the phase-plane portrait of this system. Is there more than one possiblity for the general shape and stability type of the trajectories around each of the critical points in this case? Why/why not?*

There is only one possibility for the general shape and stability type of the trajectories near each critical point since they are all structurally stable.





14. *Same instructions as **13** for the DE system* $x' = y \qquad y' = 2x - x^2$. a) Critical points: by inspection $(0,0)$ and $(2,0)$.

b) Jacobian: $J(x,y) = \begin{pmatrix} 0 & 1 \\ 2-2x & 0 \end{pmatrix}$.

Crit. pt. $(0,0)$: $\quad J(0,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

Characteristic equation: $p(\lambda) = \lambda^2 - 2 = 0$.
Eigenvalues: $\lambda = \pm\sqrt{2} \Rightarrow$ saddle.
Unstable equilibrium, structurally stable.

Crit. pt. $(2,0)$: $\quad J(2,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

Characteristic equation: $p(\lambda) = \lambda^2 + 2 = 0$.
Eigenvalues: $\pm i\sqrt{2} \Rightarrow$ center.
Stable equilibrium, not structurally stable.

c) There is only one possible type and general shape near the critical point at $(0,0)$ because it's structurally stable. But near the critical point at $(2,0)$ the non-linear system could look like a center, a spiral source or a spiral sink. By perturbing the system we got the computer program to draw all 3 possibilities.