18.03SC Differential Equations, Fall 2011
Transcript – Complex Numbers and Euler's Formula

PROFESSOR: Hi. Today we'll explore the complex numbers and Euler formula. So the first part of the problem is to write this complex number minus 2 plus 3i in polar form. And at this point, it's written in rectangular coordinate form. The second question asks you to do the reverse, to write three exponential to the i pi over 6 in rectangular coordinate form. Question c asks you to draw and label the triangle relating the rectangular coordinates to the polar coordinate form. D asks you to compute 1 over this reverse complex number that we already saw in question a. And e asks you to find the cube root of 1.

And in all these questions, you'll be using Euler formula. So why don't you pause the video, take a few minutes to work out the problem, and we'll come back.

Welcome back. So we're asked throughout the problem to go back and forth between coordinates in polar form and in rectangular form. So a key thing to remember is Euler formula from the start. So we're just going to write it up here. It allows us to express a complex exponential into the sum of its cosine plus i sine theta.

So how do we tactical question a? Question a gives us a complex number in rectangular form. So in this form, a plus i b. And we're asked to write it in polar form, which introduces the modulus of the complex number r and its phase theta. So r, modulus of the complex number that we can compute when we know its rectangular form, with its real form squared plus imaginary part squared, the whole thing under the root. So in this case, we have four plus 9. So we end up with root of 13 for the modulus of the complex number z.

So now for the phase. Using Euler formula, we can see that we can relate the rectangular form to the polar form by just introducing-- I'm going to keep r.

And you can see now that we can extract the sine and the cosine of the angle theta and relate that to a ratio of a. And the modulus r that we just found, b modulus r that we just found. Or in one move, just express it as the tangent of the angle theta, just sine over the cosine, just becomes basically b over a, which we have here. So we have the modulus r, which is now root of 13, and the angle theta that we can now extract by using the reverse of the function 10.

So just before we move to the next question, this is not one of the classical angles that you learned. So just to have an idea of where this angle lies on the trigonometric unit circle, just recall here that the sine is positive and the cosine of this angle is negative. So we're bound to be in this region, where basically theta is between pi and pi over 2. And that's the answer to question a.

So now for question b, we're asked to do the reverse, expressing the polar number 3i pi over 6 in rectangular coordinates. So now this is just a straightforward application of the Euler formula that we just saw. By just expanding the exponential, as I already wrote there. Plus i3 sine pi over 6.

And on the same trigonometric circle here, that's roughly where pi over 6 lie. And you can just re-express this as 3, root of 3 over 2, plus i, 3 over 2. So that answers the solution for question b.

So now question c. Let me just add a line here. Question c we're asked to draw and label the triangle relating rectangular to polar coordinates. So that's what we already had a sense with when we wrote this formula going from a plus i b to r exponential of i theta.

So in the complex plane, we have the real axis, an imaginary axis, and a complex number lying on this plane written in this form in a rectangular coordinate. You'd have a projection of a in the real axis, projection of value b on the imaginary axis. And in polar form this would be its modulus or distance from the origin, and its phase theta that would come in the polar form. So that's the triangle that allows us to go back and forth between the rectangular and polar coordinates.

So to almost finish, question d now asks us to compute the reverse of the original complex number that we used. So 1 over minus 2 plus 3i. So to do this, we can stay in rectangular form and basically multiply the numerator and denominator by the complex conjugate of the number.

But clearly now that we learned how to use polar coordinate expressions of this number, it's much easier to just write it directly in this form. In one step we basically arrived to the results, where we express that the angle was the reverse 10 of minus 3 over 2. And that's done.

So now are for the last question, we were asked to compute the one third root of 1. So basically, 1 to one third. So here, obviously, we're treating 1 as a complex number. And if we go in the complex plane and I just introduce here the number 1, we see that in polar form 1 is just basically a complex number with modulus 1, and angle 0 modulus 2pi. So we can write 1 as exponential 2 n pi, because it's basically angle 0 modulus 2pi.

And from here we know that we're looking at third roots, so we're going to have three roots. And these roots are going to be expressed by changing the value of n. First one, n equals to 0 is just going to give us back root of 1, because we're going to have exponential to 0 is just 1. Power of one third is just 1. n equals to 1. We are going to have exponential of 2pi over 3, which we can express again using the Euler formula, also in coordinate form. And then just write down the values.

And for the third root we take the value n equals to 2, so we have i 4 pi over 3, which again we can express as the cosine plus the sine of 4pi over 3.

So where do these roots lie? So we have root 1 for n equals to 0. The second root exponential 2pi over 3, basically in polar form would be here, where we would have the angle 2pi over 3. So 1pi over 2 would here. 2pi over 3 would be here. 3pi over 3 would be here. And 4pi over 3 is our third root, it would be here.

So then we can just write down the values. And you can do that when you know the angles, or just keep it in either form when you don't know their overall expression for the angles. So this completes the problems in all of these problems what we kept using is Euler formula to go back and forth between coordinate in rectangular form to expression of complex number in polar form. And that's the key formula that we kept using. And you'll be using this repeatedly when we will be solving other ODEs for which we can use complex number as a trick for solutions.

And this ends this session.