18.06SC Final Exam Solutions

1 (4+7=11 pts.) Suppose A is 3 by 4, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Remembering that A is 3 by 4, find its row reduced echelon form R.

(b) Find the dimensions of all four fundamental subspaces $C(A)$, $N(A)$, $C(A^{\mathrm{T}})$, $N(A^{\mathrm{T}})$.

You have enough information to find bases for one or more of these subspaces—find those bases.

Solution.

(a) Each special solution tells us the solution to $Rx = 0$ when we set one free variable $= 1$ and the others $= 0$. Here, the third and fourth variables must be the two free variables, and the other two are the pivots: $R = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now multiply out $Rx_1 = 0$ and $Rx_2 = 0$ to find the $*$'s: $R = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(The $*$'s are just the negatives of the special solutions' pivot entries.)

(b) We know the nullspace $N(A)$ has $n - r = 4 - 2 = 2$ dimensions: the special solutions x_1, x_2 form a basis.

The row space $C(A^{\mathrm{T}})$ has $r = 2$ dimensions. It's orthogonal to $N(A)$, so just pick two linearly-independent vectors orthogonal to x_1 and x_2 to form a basis: for example, $x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

(Or: $C(A^{\mathrm{T}}) = C(R^{\mathrm{T}})$ is just the row space of R, so the first two rows are a basis. Same thing!)

The column space $C(A)$ has $r = 2$ dimensions (same as $C(A^{\mathrm{T}})$). We can't write down a basis because we don't know what A is, but we can say that the first two columns of A are a basis.

The left nullspace $N(A^{\mathrm{T}})$ has $m - r = 1$ dimension; it's orthogonal to $C(A)$, so any vector orthogonal to the first two columns of A (whatever they are) will be a basis.

2 (6+3+2=11 pts.) (a) Find the inverse of a 3 by 3 upper triangular matrix U, with **nonzero** entries a, b, c, d, e, f. You could use cofactors and the formula for the inverse. Or possibly Gauss-Jordan elimination.

$$\text{Find the inverse of } \; U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

(b) Suppose the columns of U are eigenvectors of a matrix A. Show that A is also upper triangular.

(c) Explain why this U **cannot** be the same matrix as the first factor in the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$.

Solution.

(a) *By elimination:* (We keep track of the elimination matrix E on one side, and the product EU on the other. When $EU = I$, then $E = U^{-1}$.)

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & d & e & 0 & 1 & 0 \\ 0 & 0 & f & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & b/a & c/a & 1/a & 0 & 0 \\ 0 & 1 & e/d & 0 & 1/d & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/f \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & -b/ad & (be-cd)/adf \\ 0 & 1 & 0 & 0 & 1/d & -e/df \\ 0 & 0 & 1 & 0 & 0 & 1/f \end{array}\right] = \begin{bmatrix} I & U^{-1} \end{bmatrix}$$

By cofactors: (Take the minor, then "checkerboard" the signs to get the cofactor matrix, then transpose and divide by $\det(U) = adf$.)

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \rightsquigarrow \begin{bmatrix} df & 0 & 0 \\ bf & af & 0 \\ be-cd & ae & ad \end{bmatrix} \rightsquigarrow \begin{bmatrix} df & 0 & 0 \\ -bf & af & 0 \\ be-cd & -ae & ad \end{bmatrix} \rightsquigarrow \begin{bmatrix} df & -bf & be-cd \\ 0 & af & -ae \\ 0 & 0 & ad \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 1/a & -b/ad & (be-cd)/adf \\ 0 & 1/d & -e/df \\ 0 & 0 & 1/f \end{bmatrix} = U^{-1}$$

(b) We have a complete set of eigenvectors for A, so we can diagonalize: $A = U\Lambda U^{-1}$. We know U is upper-triangular, and so is the diagonal matrix Λ, and we've just shown that U^{-1} is upper-triangular too. So their product A is also upper-triangular.

(c) The columns aren't orthogonal! (For example, the product $u_1^{\mathrm{T}} u_2$ of the first two columns is $ab + 0d + 0 \cdot 0 = ab$, which is nonzero because we're assuming all the entries are nonzero.)

3 (3+3+5=11 pts.) (a) A and B are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

$$\text{rank of } A \qquad\qquad \text{rank of the block matrix } \begin{bmatrix} A & B \end{bmatrix}$$

(b) Suppose $B = A^2$. How do those ranks compare? Explain your reasoning.

(c) If A is m by n of rank r, what are the dimensions of these nullspaces?

$$\text{Nullspace of } A \qquad\qquad \text{Nullspace of } \begin{bmatrix} A & A \end{bmatrix}$$

Solution.

(a) All you can say is that rank $A \leq$ rank $[A\, B]$. (A can have any number r of pivot columns, and these will all be pivot columns for $[A\, B]$; but there could be more pivot columns among the columns of B.)

(b) Now rank $A =$ rank $[A\ A^2]$. (Every column of A^2 is a linear combination of columns of A. For instance, if we call A's first column a_1, then Aa_1 is the first column of A^2. So there are no new pivot columns in the A^2 part of $[A\ A^2]$.)

(c) The nullspace $N(A)$ has dimension $n - r$, as always. Since $[A\ A]$ only has r pivot columns — the n columns we added are all duplicates — $[A\ A]$ is an m-by-$2n$ matrix of rank r, and its nullspace $N([A\ A])$ has dimension $2n - r$.

4 (3+4+5=12 pts.) Suppose A is a 5 by 3 matrix and Ax is never zero (except when x is the zero vector).

(a) What can you say about the columns of A?

(b) Show that $A^{\mathrm{T}}Ax$ is also never zero (except when $x = 0$) by explaining this key step:

If $A^{\mathrm{T}}Ax = 0$ then obviously $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$ and then (WHY?) $Ax = 0$.

(c) We now know that $A^{\mathrm{T}}A$ is invertible. Explain why $B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is a one-sided inverse of A (which side of A?). B is NOT a 2-sided inverse of A (*explain why not*).

Solution.

(a) $N(A) = 0$ so A has full column rank $r = n = 3$: the columns are linearly independent.

(b) $x^{\mathrm{T}}A^{\mathrm{T}}Ax = (Ax)^{\mathrm{T}}Ax$ is the squared length of Ax. The only way it can be zero is if Ax has zero length (meaning $Ax = 0$).

(c) B is a left inverse of A, since $BA = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}A = I$ is the (3-by-3) identity matrix. B is not a right inverse of A, because AB is a 5-by-5 matrix but can only have rank 3. (In fact, $BA = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is the projection onto the (3-dimensional) column space of A.)

5 (5+5=10 pts.) If A is 3 by 3 symmetric positive definite, then $Aq_i = \lambda_i q_i$ with positive eigenvalues and orthonormal eigenvectors q_i.

Suppose $x = c_1 q_1 + c_2 q_2 + c_3 q_3$.

(a) Compute $x^{\mathrm{T}}x$ and also $x^{\mathrm{T}}Ax$ in terms of the c's and λ's.

(b) Looking at the ratio of $x^{\mathrm{T}}Ax$ in part (a) divided by $x^{\mathrm{T}}x$ in part (a), what c's would make that ratio as large as possible? You can assume $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Conclusion: the ratio $x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ is a maximum when x is ______.

Solution.

(a)

$$\begin{aligned} x^{\mathrm{T}}x &= (c_1 q_1^{\mathrm{T}} + c_2 q_2^{\mathrm{T}} + c_3 q_3^{\mathrm{T}})(c_1 q_1 + c_2 q_2 + c_3 q_3) \\ &= c_1^2 q_1^{\mathrm{T}} q_1 + c_1 c_2 q_1^{\mathrm{T}} q_2 + \cdots + c_3 c_2 q_3^{\mathrm{T}} q_2 + c_3^2 q_3^{\mathrm{T}} q_3 \\ &= c_1^2 + c_2^2 + c_3^2. \end{aligned}$$

$$\begin{aligned} x^{\mathrm{T}}Ax &= (c_1 q_1^{\mathrm{T}} + c_2 q_2^{\mathrm{T}} + c_3 q_3^{\mathrm{T}})(c_1 A q_1 + c_2 A q_2 + c_3 A q_3) \\ &= (c_1 q_1^{\mathrm{T}} + c_2 q_2^{\mathrm{T}} + c_3 q_3^{\mathrm{T}})(c_1 \lambda_1 q_1 + c_2 \lambda_2 q_2 + c_3 \lambda_3 q_3) \\ &= c_1^2 \lambda_1 q_1^{\mathrm{T}} q_1 + c_1 c_2 \lambda_2 q_1^{\mathrm{T}} q_2 + \cdots + c_3 c_2 \lambda_2 q_3^{\mathrm{T}} q_2 + c_3^2 \lambda_3 q_3^{\mathrm{T}} q_3 \\ &= c_1^2 \lambda_1 + c_2^2 \lambda_2 + c_3^2 \lambda_3. \end{aligned}$$

(b) We maximize $(c_1^2\lambda_1 + c_2^2\lambda_2 + c_3^2\lambda_3)/(c_1^2 + c_2^2 + c_3^2)$ when $c_1 = c_2 = 0$, so $x = c_3 q_3$ is a multiple of the eigenvector q_3 with the largest eigenvalue λ_3.

(Also notice that the maximum value of this "Rayleigh quotient" $x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ is the largest eigenvalue itself. This is another way of finding eigenvectors: maximize $x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ numerically.)

6 (4+4+4=12 pts.) (a) Find a linear combination w of the linearly independent vectors v and u that is perpendicular to u.

(b) For the 2-column matrix $A = \begin{bmatrix} u & v \end{bmatrix}$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.

(c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v.

Solution.

(a) You could just write down $w = 0u + 0v = 0$ — that's perpendicular to everything! But a more useful choice is to subtract off just enough u so that $w = v - cu$ is perpendicular to u. That means $0 = w^{\mathrm{T}}u = v^{\mathrm{T}}u - cu^{\mathrm{T}}u$, so $c = (v^{\mathrm{T}}u)/(u^{\mathrm{T}}u)$ and

$$w = v - (\frac{v^{\mathrm{T}}u}{u^{\mathrm{T}}u})u.$$

(b) We already know u and w are orthogonal; just normalize them! Take $q_1 = u/|u|$ and $q_2 = w/|w|$. Then solve for the columns r_1, r_2 of R: $Qr_1 = u$ so $r_1 = \begin{bmatrix} |u| \\ 0 \end{bmatrix}$, and $Qr_2 = v$ so $r_2 = \begin{bmatrix} c|u| \\ |w| \end{bmatrix}$. (Where $c = (v^{\mathrm{T}}u)/(u^{\mathrm{T}}u)$ as before.)

Then $Q = [q_1\ q_2]$ and $R = [r_1\ r_2]$.

(c) $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = (QR)(R^{\mathrm{T}}Q^{\mathrm{T}}QR)^{-1}(R^{\mathrm{T}}Q^{\mathrm{T}}) = (QR)(R^{\mathrm{T}}Q^{\mathrm{T}}) = \underline{QQ^{\mathrm{T}}}$.

7 (4+3+4=11 pts.) (a) Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?

(c) Find the determinants of C and $C + I$ and $C + 2I$.

Solution.

(a) Take the determinant of $C - \lambda I$ (I expanded by cofactors): $\lambda^4 - 1 = 0$. The roots of this "characteristic equation" are the eigenvalues: $+1, -1, i, -i$.

The eigenvalues of C^2 are just $\lambda^2 = \pm 1$ (two of each).

(Here's a "guessing" approach. Since $C^4 = I$, all the eigenvalues λ^4 of C^4 are 1: so $\lambda = 1, -1, i, -i$ are the only possibilities. Just check to see which ones work. Then the eigenvalues of C^2 must be ± 1.)

(b) For any permutation matrix, $C^{-1} = C^{\mathrm{T}}$: so

$$C^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $(C^2)^{-1} = C^2$ is itself.

(c) The determinant of C is the product of its eigenvalues: $1(-1)i(-i) = \underline{-1}$.

Add 1 to every eigenvalue to get the eigenvalues of $C + I$ *(if $C = S\Lambda S^{-1}$, then $C + I = S(\Lambda + I)S^{-1}$)*: $2(0)(1+i)(1-i) = \underline{0}$.

(Or let $\lambda = -1$ in the characteristic equation $\det(C - \lambda I)$*.)*

Add 2 to get the eigenvalues of $C + 2I$ *(or let $\lambda = -2$)*: $3(1)(2+i)(2-i) = \underline{15}$.

8 (4+3+4=11 pts.) Suppose a rectangular matrix A has independent columns.

(a) How do you find the best least squares solution $\widehat{x}$ to $Ax = b$? By taking those steps, give me a formula (letters not numbers) for $\widehat{x}$ and also for $p = A\widehat{x}$.

(b) The projection p is in which fundamental subspace associated with A? The error vector $e = b - p$ is in which fundamental subspace?

(c) Find by any method the projection matrix P onto the column space of A:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

Solution.

(a)

$$\begin{aligned} & & Ax &= b \\ \text{Least-squares "solution":} & & A^{\mathrm{T}}A\hat{x} &= A^{\mathrm{T}}b \\ A^{\mathrm{T}}A \text{ is invertible:} & & \hat{x} &= (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b \\ \text{and } p = A\hat{x} \text{ is:} & & A\hat{x} &= A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b \end{aligned}$$

(b) $p = A\hat{x}$ is a linear combination of columns of A, so it's in the column space $C(A)$. The error $e = b - p$ is orthogonal to this space, so it's in the left nullspace $N(A^{\mathrm{T}})$.

(c) I used $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. Since $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, its inverse is $\begin{bmatrix} 1/10 & 0 \\ 0 & 1/10 \end{bmatrix} = \frac{1}{10}I$, and

$$P = \frac{1}{10}\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

9 (3+4+4=11 pts.) This question is about the matrices with 3's on the main diagonal, 2's on the diagonal above, 1's on the diagonal below.

$$A_1 = \begin{bmatrix} 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A_n = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}$$

(a) What are the determinants of A_2 and A_3?

(b) The determinant of A_n is D_n. Use cofactors of row 1 and column 1 to find the numbers a and b in the recursive formula for D_n:

$$(*) \qquad D_n = a\, D_{n-1} + b\, D_{n-2}\,.$$

(c) This equation $(*)$ is the same as

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}.$$

>From the eigenvalues of that matrix, how fast do the determinants D_n grow? (If you didn't find a and b, say how you would answer part (c) for any a and b) For 1 point, find D_5.

Solution.

(a) $\det(A_2) = 3 \cdot 3 - 1 \cdot 2 = 7$ and $\det(A_3) = 3\det(A_2) - 2 \cdot 1 \cdot 3 = 15$.

(b) $D_n = 3D_{n-1} + (-2)D_{n-2}$. *(Show your work.)*

(c) The trace of that matrix A is $a = 3$, and the determinant is $-b = 2$. So the characteristic equation of A is $\lambda^2 - a\lambda - b = 0$, which has roots (the eigenvalues of A)

$$\lambda_\pm = \frac{a \pm \sqrt{a^2 - 4(-b)}}{2} = \frac{3 \pm 1}{2} \quad = 1 \text{ or } 2.$$

D_n grows at the same rate as the largest eigenvalue of A^n, $\lambda_+^n = 2^n$.

The final point: $D_5 = 3D_4 + 2D_3 = 3(3D_3 + 2D_2) + 2D_3 = 11D_3 + 6D_2 = 207$.