5.11 **Complete Fourier series**

58. Expand each of the following functions in a Fourier series of period equal to the length of the indicated interval of representation:

$$\begin{array}{ll} (a)\ f(x) = a + bx & (0 < x < P), \\ (b)\ f(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x > 0) \end{cases} & (-1 < x < 1), \\ (c)\ f(x) = \sin x & (0 \le x \le \pi), \\ (d)\ f(x) = x & (1 < x < 2). \end{array}$$

Solution. In this problem, we need to expand a function $f(x)$ in a complete Fourier series in (a, b); the periodic extension of $f(x)$ has period equal to the length $L = b - a$ of the interval. With

$$f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos \frac{n\pi x}{L/2} + B_n \sin \frac{n\pi x}{L/2}],$$

the coefficients are given by the formulas

$$\begin{aligned} A_0 &= \frac{1}{L} \int_a^b dx\, f(x), \\ A_{n\ge 1} &= \frac{2}{L} \int_a^b dx\, f(x)\, \cos \frac{2n\pi x}{L}, \\ B_{n\ge 1} &= \frac{2}{L} \int_a^b dx\, f(x)\, \sin \frac{2n\pi x}{L}, \end{aligned}$$

which are direct generalizations of the formulas derived in class for the *symmetric* interval $(-l, l)$ where $L = 2l$.

(a) Let $f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{2n\pi x}{P} + B_n \sin \frac{2n\pi x}{P} \right)$ in $(0, P)$; then,

$$
\begin{aligned}
A_0 &= \frac{1}{P} \int_0^P (a + bx) dx \\
&= a + \frac{bP}{2}, \quad \text{and, for } n \geq 1, \\
A_{n\geq 1} &= \frac{2}{P} \int_0^P (a + bx) \cos \frac{2n\pi x}{P} \, dx \\
&= \frac{1}{n\pi} \int_0^P (a + bx) \; d \left(\sin \frac{2n\pi x}{P} \right) \\
&= \frac{1}{n\pi} \left((a + bx) \sin \frac{2n\pi x}{P} \bigg|_0^P - \int_0^P b \sin \frac{2n\pi x}{P} \, dx \right) \\
&= \frac{bP}{2n^2\pi^2} \int_0^P d \left(\cos \frac{2n\pi x}{P} \right) \\
&= 0, \\
B_{n\geq 1} &= \frac{2}{P} \int_0^P (a + bx) \sin \frac{2n\pi x}{P} \, dx \\
&= -\frac{1}{n\pi} \int_0^P (a + bx) \; d \left(\cos \frac{2n\pi x}{P} \right) \\
&= -\frac{1}{n\pi} \left((a + bx) \cos \frac{2n\pi x}{P} \bigg|_0^P - \int_0^P b \cos \frac{2n\pi x}{P} \, dx \right) \\
&= -\frac{bP}{n\pi}.
\end{aligned}
$$

It follows that

$$
f(x) = a + \frac{bP}{2} - \frac{bP}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{P} \qquad \text{in } (0, P).
$$

(b) Let $f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\pi x) + B_n \sin(n\pi x)]$ in $(-1, 1)$; then,

$$\begin{aligned}
A_0 &= \frac{1}{2}\int_{-1}^{1} f(x)\ dx \\
&= \frac{1}{2}\int_{0}^{1} dx \\
&= \frac{1}{2}, \\
A_{n\geq 1} &= \int_0^1 \cos(n\pi x)\ dx = 0, \\
B_{n\geq 1} &= \int_0^1 \sin(n\pi x)\ dx \\
&= -\frac{1}{n\pi}\int_0^1 d[\cos(n\pi x)] \\
&= \frac{1}{n\pi}[1 - \cos(n\pi)] \\
&= \frac{1}{n\pi}[1 - (-1)^n] \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

So,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin[(2n-1)\pi x]\right).$$

(c) Let $f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(2nx) + B_n \sin(2nx)]$ in $[0, \pi]$; then,

$$\begin{aligned}
A_0 &= \frac{1}{\pi}\int_0^{\pi} \sin x\ dx = \frac{2}{\pi}, \\
A_{n\geq 1} &= \frac{2}{\pi}\int_0^{\pi} \sin x\ \cos(2nx)\ dx \\
&= \frac{1}{\pi}\int_0^{\pi} \{\sin[(2n+1)x] - \sin[(2n-1)x]\}\ dx \\
&= \frac{1}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) \\
&= -\frac{4}{\pi(4n^2-1)}, \\
B_{n\geq 1} &= \frac{2}{\pi}\int_0^{\pi} \sin x\ \sin(2nx)\ dx \\
&= \frac{1}{\pi}\int_0^{\pi} \{\cos[(2n-1)x] - \cos[(2n+1)x]\}\ dx = 0.
\end{aligned}$$

So,

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cdot \cos(2nx)\right) \quad \text{in } [0,\pi].$$

(d) Let $f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(2n\pi x) + B_n \sin(2n\pi x)]$ in $(1, 2)$; then,

$$\begin{aligned}
A_0 &= \int_1^2 x\ dx = \frac{3}{2},\\
A_{n\geq 1} &= 2\int_1^2 x\cos(2n\pi x)\ dx\\
&= \frac{1}{n\pi}\int_1^2 x\ d(\sin(2n\pi x))\\
&= \frac{1}{n\pi}\left(x\sin(2n\pi x)|_1^2 - \int_1^2 \sin(2n\pi x)\ dx\right) = 0,\\
B_{n\geq 1} &= 2\int_1^2 x\sin(2n\pi x)\ dx\\
&= -\frac{1}{n\pi}\int_1^2 x\ d(\cos(2n\pi x))\\
&= -\frac{1}{n\pi}\left(x\cos(2n\pi x)|_1^2 - \int_1^2 \cos(2n\pi x)\ dx\right) = -\frac{1}{n\pi}.
\end{aligned}$$

So

$$f(x) = \frac{3}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(2n\pi x) \quad \text{in } (1,2).$$

60. Obtain the Fourier series of period 2π which represents the solution of the problem

$$\frac{d^2y}{dx^2} + \Lambda y = h(x),\ \ y(-\pi) = y(\pi),\ \ y'(-\pi) = y'(\pi)$$

when

$$h(x) = \begin{cases} 0 & (-\pi < x < 0),\\ 1 & (0 < x < \frac{\pi}{2}),\\ 0 & (\frac{\pi}{2} < x < \pi),\end{cases}$$

assuming that $\Lambda \neq p^2$ $(p = 0, 1, 2, \ldots)$.

Solution. Let $h(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$ in $[-\pi, \pi]$; then,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} dx = \frac{1}{4}, \\ a_{n\geq 1} &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(nx)\ dx = \frac{1}{n\pi}\sin\frac{n\pi}{2}, \\ b_{n\geq 1} &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin(nx)\ dx = \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right). \end{aligned}$$

So,

$$h(x) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\frac{n\pi}{2}\cos(nx) + \left(1-\cos\frac{n\pi}{2}\right)\sin(nx)\right].$$

Let $y = A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$ be the solution of $\frac{d^2y}{dx^2} + \Lambda y = h(x),\ y(-\pi) = y(\pi),\ y'(-\pi) = y'(\pi)$. We determine A_n and B_n as follows:

$$\begin{aligned} h(x) &= \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\frac{n\pi}{2}\cos(nx) + \left(1-\cos\frac{n\pi}{2}\right)\sin(nx)\right] \\ &= \frac{d^2y}{dx^2} + \Lambda y \\ &= \Lambda A_0 + \sum_{n=1}^{\infty}(A-n^2)[A_n\cos(nx) + B_n\sin(nx)], \end{aligned}$$

by differentiating the Fourier series term by term. This relation gives $A_0 = \frac{1}{4\Lambda}$, and, for $n \geq 1$, $A_{n\geq 1} = \frac{\sin\frac{n\pi}{2}}{n\pi(\Lambda-n^2)}$, $B_{n\geq 1} = \frac{1-\cos\frac{n\pi}{2}}{n\pi(\Lambda-n^2)}$. Thus, the solution is

$$y = \frac{1}{4\Lambda} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}\cos(nx) + \left(1-\cos\frac{n\pi}{2}\right)\sin(nx)}{n(\Lambda-n^2)} \quad \text{in } [-\pi,\pi].$$

61. a) If the representation

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} \qquad (0 \leq x \leq l)$$

is valid, show formally that

$$\frac{2}{l}\int_0^l (f(x))^2\ dx = 2A_0^2 + \sum_{n=1}^{\infty} A_n^2.$$

Solution: Clearly,

$$\begin{aligned}\int_0^l (f(x))^2\ dx &= \int_0^l \left(\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l}\right)^2\ dx \\ &= \sum_{m,n=0}^{\infty} A_n A_m \left(\int_0^l \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\ dx\right).\end{aligned}$$

But

$$\int_0^l \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\ dx = \begin{cases} 0, & \text{if } m \neq n \\ l, & \text{if } m = n = 0 \\ l/2, & \text{if } m = n \geq 1. \end{cases}$$

It follows that

$$\int_0^l (f(x))^2\ dx = l \cdot A_0^2 + \frac{l}{2}\sum_{n=1}^{\infty} A_n^2.$$

Hence,

$$\frac{2}{l}\int_0^l (f(x))^2\ dx = 2A_0^2 + \sum_{n=1}^{\infty} A_n^2.$$

b) If the representation

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \qquad (0 < x < l)$$

is valid, show formally that

$$\frac{2}{l}\int_0^l (f(x))^2\ dx = \sum_{n=1}^{\infty} B_n^2$$

Solution:

$$\begin{aligned}\int_0^l (f(x))^2\ dx &= \int_0^l \left(\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\right)^2\ dx \\ &= \sum_{m,n=1}^{\infty} B_m B_n \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\ dx.\end{aligned}$$

But

$$\int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\ dx = \begin{cases} \frac{l}{2} & m = n \\ 0 & m \neq n \end{cases}$$

This implies that

$$\int_0^l (f(x))^2\ dx = \frac{l}{2}\sum_{n=1}^{\infty} B_n^2.$$

Hence,

$$\frac{2}{l}\int_0^l (f(x))^2\ dx = \sum_{n=1}^{\infty} B_n^2.$$

c) If the representation

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right) \qquad (-l < x < l)$$

is valid, show formally that

$$\frac{1}{l} \int_{-l}^{l} (f(x))^2 \ dx = 2A_0^2 + \sum_{n=1}^{\infty} (A_n^2 + B_n^2).$$

Solution:

$$\begin{aligned} \int_{-l}^{l} (f(x))^2 \ dx &= \int_{-l}^{l} \left(A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right) \right)^2 \ dx \\ &= \sum_{m,n=0}^{\infty} A_m A_n \int_{-l}^{l} \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l} \ dx \\ &+ \sum_{m,n=1}^{\infty} B_m B_n \int_{-l}^{l} \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \ dx \\ &+ 2 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_m B_n \int_{-l}^{l} \cos \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \ dx \end{aligned}$$

By virtue of

$$\int_{-l}^{l} \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l} \ dx = \int_{-l}^{l} \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \ dx = 0 \text{ for } m \neq n,$$

$$\int_{-l}^{l} \cos \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \ dx = 0, \quad \text{for all } m, n,$$

$$\int_{-l}^{l} dx = 2l,$$

and

$$\int_{-l}^{l} \left(\cos \frac{n\pi x}{l} \right)^2 \ dx = \int_{-l}^{l} \left(\sin \frac{n\pi x}{l} \right)^2 \ dx = l, \quad n \geq 1,$$

we find that

$$\int_{-l}^{l} (f(x))^2 \ dx = 2lA_0^2 + l \sum_{n=1}^{\infty} (A_n^2 + B_n^2).$$

Hence,

$$\frac{1}{l} \int_{-l}^{l} (f(x))^2 \ dx = 2A_0^2 + \sum_{n=1}^{\infty} (A_n^2 + B_n^2).$$