SOLUTION SET III FOR 18.075–FALL 2004

10. Functions of a Complex Variable

10.7. **Taylor Series.** .

48. Obtain each of the following series expansions by any convenient method:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ... = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!} \qquad (|z| < \infty), \tag{1}$$

$$\frac{\cosh z - 1}{z^2} = \frac{1}{2!} + \frac{z^2}{4!} + \frac{z^4}{6!} + ... = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+2)!} \qquad (|z| < \infty), \tag{2}$$

$$\frac{e^z}{1-z} = 1 + 2z + \frac{5}{2}z^2 + \frac{8}{3}z^3 + ... \qquad (|z| < 1), \tag{3}$$

$$\frac{a^2}{z^2} = 1 + 2\frac{z+a}{a} + 3\frac{(z+a)^2}{a^2} + ... \qquad (|z+a| < |a|). \tag{4}$$

Solution. (a) We repeat what we did in class. For $|z| < \infty$,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... = z(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ...)$$

Since the series in parenthesis is absolutely convergent (by the same criterion used to prove the absolute convergence of the Taylor series of e^{iz} and $\sin z$) we can divide by z both terms of the above equality. Thus we get

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ... = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}.$$

From the uniqueness of the Taylor series we have that the series on the right hand side is the Taylor series of $\sin z/z$.

(b) From the formulas $\cos z = (e^{iz} + e^{-iz})/2$ and $\cosh z = \cos(iz)$, we get for $|z| < \infty$:

$$\cosh z - 1 = \frac{z^2}{2!} + \frac{z^4}{4!} - ... = z^2(\frac{1}{2!} + \frac{z^2}{4!} + ...)$$

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Since the series in parenthesis is absolutely convergent (by the same criterion used to prove the absolute convergence of the Taylor series of e^z and $\cosh z$) we can divide by z^2 both terms of the above equality. Thus we get

$$\frac{\cosh z - 1}{z^2} = \frac{1}{2!} + \frac{z^2}{4!} + \frac{z^4}{6!} + ... = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+2)!}$$

From the uniqueness of the Taylor series we have that the series on the right hand side is the Taylor series of $(\cosh z - 1)/z^2$.

(c)From the definition of e^z we have that, for $|z| < \infty$,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...,$$

while, for $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + ...$$

Since the above series are absolutely convergent, we can multiply them term by term and we obtain a series which converges absolutely to the product $e^z/(1-z)$ in the disk $|z| < 1$. Therefore, in $|z| < 1$, we have

$$\frac{e^z}{1-z} = (1 + z + z^2 + ...) + (z + z^2 + z^3 + ...) + \frac{1}{2!}(z^2 + z^3 + ...) = 1 + 2z + \frac{5}{2}z^2 + ...$$

From the uniqueness of the Taylor series, the series on the right hand side is the Taylor series of $e^z/(1-z)$.

(d) Observe that $a^2/z^2 = 1/[1 - (z+a)/a]^2$. Therefore, using the geometric series $1/(1-w) = \sum_{n=0}^{\infty} w^n, |w| < 1$ and differentiating in w term by term as $1/(1-w)^2 = (d/dw)[1/(1-w)] = \sum_{n=0}^{\infty}(n+1)\, w^n,\ |w| < 1$ for $w = (z+a)/a$, we get:

$$\frac{a^2}{z^2} = \sum_{n=0}^{\infty}(n+1)\ \left(\frac{z+a}{a}\right)^n, \qquad |z+a| < |a|.$$

Then

$$\frac{a^2}{z^2} = 1 + 2\frac{z+a}{a} + 3\frac{(z+a)^2}{a^2} + ... \qquad (|z+a| < |a|).$$

10.8. **Laurent Series.** .

51. Expand the function $f(z) = 1/(1-z)$ in each of the following series:

(a) a Taylor series of powers of z for $|z| < 1$;

(b) a Laurent series of powers of z for $|z| > 1$;

(c) a Taylor series of powers of $z+1$ for $|z+1| < 2$, by first writing $f(z) = [2-(z+1)]^{-1} = \frac{1}{2}[1-(z+1)/2]^{-1}$;

(d) a Laurent series of powers of $z+1$ for $|z+1| > 2$, by first writing $f(z) = -[1/(z+1)]/[1-2/(z+1)]$.

Solution. (a)In the disk $|z| < 1$, we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

which is the familiar geometric series.

(b)In $|z| > 1$ we have,

$$\begin{aligned}\frac{1}{1-z} &= \frac{1/z}{1/z-1} = -\frac{1}{z}\frac{1}{(1-1/z)} = \\ &= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} \\ &= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - .....\end{aligned}$$

where we have used the geometric series expansion for $1/(1-1/z)$ in $|1/z| < 1$ or, equivalently, $|z| > 1$.

(c)Using the geometric series expansion for the function $1/[1-(z+1)/2]$ in $|(z+1)/2| < 1$ or, equivalently, $|z+1| < 2$, we have:

$$\frac{1}{1-z} = \frac{1}{2}\frac{1}{[1-(z+1)/2]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(z+1)^n}{2^{n+1}}.$$

(d)Using the geometric series expansion for the function $1/[1-2/(z+1)]$ in $|2/(z+1)| < 1$ or, equivalently, $|z+1| > 2$, we get:

$$\frac{1}{1-z} = -\frac{1}{z+1}\frac{1}{[1-2/(z+1)]} = -\frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{2}{z+1}\right)^n = -\sum_{n=0}^{\infty}\frac{2^n}{(z+1)^{n+1}}.$$

52. Expand the function $f(z) = 1/[z(1-z)]$ in a Laurent (or Taylor)series which converges in each of the following regions:

(a) $0 < |z| < 1$, (b) $|z| > 1$,
(c) $0 < |z-1| < 1$, (d) $|z-1| > 1$,
(e) $|z+1| < 1$, (f) $1 < |z+1| < 2$,
(g) $|z+1| > 2$.

Solution.(a)Using the geometric series expansion for $1/(1-z)$ in $0 < |z| < 1$ we get:

$$\begin{aligned}\frac{1}{z(1-z)} &= \frac{1}{z}\sum_{n=0}^{\infty} z^n = \sum_{n=-1}^{\infty} z^n \\ &= \frac{1}{z} + 1 + z + z^2 + ...\end{aligned}$$

(b)Using part (b) of exercise 51 we get, in $|z| > 1$,

$$\begin{aligned}\frac{1}{z(1-z)} &= -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+2}} \\ &= -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \ldots\end{aligned}$$

(c)Using the geometric series expansion for $1/[1-(1-z)]$ in $0 < |z-1| < 1$, we get:

$$\begin{aligned}\frac{1}{z(1-z)} &= \frac{1}{(1-(1-z))}\frac{1}{(1-z)} = \frac{1}{1-z}\sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(1-z)^{n-1} \\ &= \frac{1}{1-z} + 1 + (1-z) + (1-z)^2 + \ldots\end{aligned}$$

(d)Using the geometric series expansion for $1/[1-1/(1-z)]$ in $1/|z-1| < 1$ or, equivalently, $|z-1| > 1$, we get:

$$\begin{aligned}\frac{1}{z(1-z)} &= -\frac{1}{[1-1/(1-z)]}\frac{1}{(1-z)^2} = -\frac{1}{(1-z)^2}\sum_{n=0}^{\infty}\frac{1}{(1-z)^n} = -\sum_{n=0}^{\infty}\frac{1}{(1-z)^{n+2}} \\ &= -\frac{1}{(1-z)^2} - \frac{1}{(1-z)^3} - \frac{1}{(1-z)^4} + \ldots\end{aligned}$$

(e)Using the geometric series expansion for $1/[1-(z+1)]$ in $|z+1| < 1$ and the one for $1/[1-(z+1)/2]$ in $|z+1| < 2$, we get in $|z+1| < 1$ (overlap region of the two disks of convergence):

$$\begin{aligned}\frac{1}{z(1-z)} &= \frac{-1}{1-(z+1)} + \frac{1}{2}\frac{1}{(1-(z+1)/2)} = -\sum_{n=0}^{\infty}(z+1)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n = \\ &= -\sum_{n=0}^{\infty}\left[1-\left(\frac{1}{2}\right)^{n+1}\right](z+1)^n = -\frac{1}{2} - \frac{3}{4}(z+1) - \frac{7}{8}(z+1)^2 + \ldots\end{aligned}$$

(f)Using the geometric series expansion for $1/(1-1/(z+1))$ in $|z+1| > 1$ and the one for $1/(1-(z+1)/2)$ in $|z+1| < 2$, we get in $1 < |z+1| < 2$ (intersection of the two disks of convergence):

$$\begin{aligned}\frac{1}{z(1-z)} &= \frac{1}{(z+1)}\frac{1}{(1-1/(z+1))} + \frac{1}{2}\frac{1}{(1-(z+1)/2)} = \\ &= \frac{1}{z+1}\sum_{n=0}^{\infty}\frac{1}{(z+1)^n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n = \\ &= \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \frac{1}{2}\sum_{n=0}^{\infty}(\frac{z+1}{2})^n\end{aligned}$$

(g)Using the geometric series expansion for $1/[1-1/(z+1)]$ in $|z+1| > 1$ and the one for $1/(1-2/(z+1))$ in $|z+1| > 2$, we get in $|z+1| > 2$ (intersection of the two disks of convergence):

$$\begin{aligned}\frac{1}{z(1-z)} &= \frac{1}{(z+1)}\frac{1}{(1-1/(z+1))} - \frac{1}{(z+1)}\frac{1}{(1-2/(z+1))} = \\ &= \frac{1}{z+1}\sum_{n=0}^{\infty}\frac{1}{(z+1)^n} - \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{2}{z+1}\right)^n = \\ &= \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{(z+1)^{n+1}} = \\ &= -\sum_{n=0}^{\infty}(2^n-1)\frac{1}{(z+1)^{n+1}} = -\frac{1}{(z+1)^2} - 3\frac{1}{(z+1)^3} - \ldots\end{aligned}$$

10.9. **Singularities of Analytic Functions.** .

61. Locate and classify the singularities of the following functions:
(a) $\frac{z}{z^2+1}$, (b) $\frac{1}{z^3+1}$, (c) $\log(z^2+1)$, (d) $(z^2-3z+2)^{\frac{2}{3}}$, (e) $\tan z$, (f) $\tan^{-1}(z-1)$.

Solution. (a)We have

$$\frac{z}{z^2+1} = \frac{1}{2}\left(\frac{1}{z-i} + \frac{1}{z+i}\right).$$

Then $\frac{z}{z^2+1}$ has a simple pole at $z+i=0$, i.e., $z=-i$, since $(z+i)\frac{z}{z^2+1} = \frac{z}{z-i}$ is analytic at $z=-i$ with value $\neq 0$. Similarly, $\frac{z}{z^2+1}$ has a simple pole at $z-i=0$, i.e., $z=i$, since $\frac{1}{z+i}$ is analytic at $z=i$ with value $\neq 0$.

(b) We have

$$\frac{1}{z^3+1} = \frac{1}{z+1} \cdot \frac{1}{z-(\frac{1+i\sqrt{3}}{2})} \cdot \frac{1}{z-(\frac{1-i\sqrt{3}}{2})}.$$

So $\frac{1}{z^3+1}$ has simple poles at $z=-1$, $z=\frac{1+i\sqrt{3}}{2}$, and $z=\frac{1-i\sqrt{3}}{2}$.

(c) We have

$$\log(z^2+1) = \log[(z+i)(z-i)] = \log(z+i) + \log(z-i),$$

where we possibly add integral multiples of $2\pi i$ to the right-hand side. We see that $\log(z+i)$ in the right-hand side has a branch point at $z=-i$, and $\log(z-i)$ is analytic at $z=-i$. So $\log(z^2+1)$ has a branch point at $z=-i$. Similarly, since $\log(z-i)$ has a branch point at $z=i$ and $\log(z+i)$ is analytic at $z=i$, $\log(z^2+1)$ has a branch point at $z=i$.

(d) We have

$$(z^2-3z+2)^{\frac{2}{3}} = [(z-2)(z-1)]^{2/3} = w^{2/3}, \quad w=(z-2)(z-1).$$

So $(z^2-3z+2)^{\frac{2}{3}}$ has branch points at $w=(z-2)(z-1)=0$, i.e., at $z=2$ and $z=1$.

(e) We have

$$\tan z = \frac{\sin z}{\cos z}.$$

So $\tan z$ has poles at $\cos z = 0$. Hence, the singularities of $\tan z$ are $z = z_n = n\pi + \frac{\pi}{2}$, where n:integer, and each of these singularities is a pole. Note that $(z - z_n)\tan z$ is analytic in a vicinity of $z = z_n$. In particular,

$$\lim_{z\to n\pi+\frac{\pi}{2}} [z - (n\pi + \frac{\pi}{2})] \tan z = -1.$$

It follows that all these poles are simple.

(f) First, we find a formula for $\tan^{-1} z$ (check with p. 550 of textbook). Let $w = \tan^{-1} z$. Then $z = \tan w = \frac{\sin w}{\cos w} = -i\frac{e^{2iw}-1}{e^{2iw}+1}$. By solving with respect to e^{2iw} we get $e^{2iw} = \frac{1+iz}{1-iz}$ or

$$w = \tan^{-1} z = \frac{1}{2i} \log\left(\frac{z-i}{z+i}\right) = \frac{1}{2i}[\log(z-i) - \log(z+i)],$$

with the possible addition of integral multiples of $2\pi i$ to the right-hand side. It follows that $\tan^{-1} z$ has branch points at $z = \pm i$. Hence, $\tan^{-1}(z-1)$ has branch points at $z - 1 = \pm i$, or $z = 1 \pm i$.

62. Show that the function

$$f(z) = \frac{\cosh z - 1}{\sinh z - z} \tag{5}$$

has a simple pole at the origin.

Proof. Clearly,

$$\cosh z - 1 = \sum_{n=1}^{\infty} \frac{z^{2n}}{(2n)!} = z^2 \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+2)!},$$

and

$$\sinh z - z = \sum_{n=1}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z^3 \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+3)!}.$$

Let

$$g(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+2)!},$$

and

$$h(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+3)!}.$$

Then both $g(z)$ and $h(z)$ are analytic functions in the complex plane (set **C**) and have nonzero values at the origin, $g(0) \neq 0 \neq h(0)$. It's clear that $\cosh z - 1 = z^2 g(z)$, and $\sinh z - z = z^3 h(z)$. So $f(z) = \frac{1}{z} \frac{g(z)}{h(z)}$. Thus, $f(z)$ has a simple pole at the origin.

10.12. **Residues.** .

78. Calculate the residues of the following functions at each of the poles in the finite part of the plane:
(a) $\frac{e^z}{z^2+a^2}$, (b) $\frac{1}{z^4-a^4}$, (c) $\frac{\sin z}{z^2}$, (d) $\frac{\sin z}{z^3}$, (e) $\frac{1+z^2}{z(z-1)^2}$, (f) $\frac{1}{(z^2+a^2)^2}$, (g) $\frac{e^{az}}{2z^2-5z+2}$, (h) $\frac{e^{z-1}-1}{1-z^2}$, (i) $\frac{1-\cos az}{z^9}$, (j) $\frac{\sinh z}{\cosh z-1}$, (k) $\frac{z}{\sin^2 z}$, (l) $\frac{(1-\cos z)^2}{z^7}$.

Solution. In the following, we use the notation

$$Res(a) = Res_{z=a}[f(z)] \equiv Res[f(z), a].$$

(a) We have the formula

$$\frac{e^z}{z^2+a^2} = \frac{e^z}{(z+ai)(z-ai)}.$$

So $\frac{e^z}{z^2+a^2}$ has simple poles at $z = \pm ai$.

$$Res[\frac{e^z}{z^2+a^2}, ai] = \frac{e^z}{z+ai}|_{z=ai} = \frac{e^{ai}}{2ai},$$
$$Res[\frac{e^z}{z^2+a^2}, -ai] = \frac{e^z}{z-ai}|_{z=-ai} = -\frac{e^{-ai}}{2ai}.$$

(b) We have

$$\frac{1}{z^4-a^4} = \frac{1}{(z+a)(z-a)(z+ai)(z-ai)}.$$

So $\frac{1}{z^4-a^4}$ has simple poles at $\pm a$, $\pm ai$.

$$Res[\frac{1}{z^4-a^4}, a] = \frac{1}{(z+a)(z+ai)(z-ai)}|_{z=a} = \frac{1}{4a^3},$$
$$Res[\frac{1}{z^4-a^4}, -a] = \frac{1}{(z-a)(z+ai)(z-ai)}|_{z=-a} = -\frac{1}{4a^3},$$
$$Res[\frac{1}{z^4-a^4}, ai] = \frac{1}{(z+a)(z-a)(z+ai)}|_{z=ai} = -\frac{1}{4ia^3},$$
$$Res[\frac{1}{z^4-a^4}, -ai] = \frac{1}{(z+a)(z-a)(z-ai)}|_{z=-ai} = \frac{1}{4ia^3}.$$

(c) We have

$$\frac{\sin z}{z^2} = \frac{1}{z} \cdot \frac{\sin z}{z}.$$

So $\frac{\sin z}{z^2}$ has a simple pole at the origin, and

$$Res[\frac{\sin z}{z^2}, 0] = \frac{\sin z}{z}|_{z=0} = 1.$$

(d) We have

$$\frac{\sin z}{z^3} = \frac{1}{z^2} \cdot \frac{\sin z}{z}.$$

So $\frac{\sin z}{z^3}$ has a pole of order 2 (i.e., double pole) at the origin, and

$$\frac{\sin z}{z^3} = \frac{1}{z^3}\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n-2}}{(2n+1)!}.$$

Note that the right-hand side has a z^{-2} term but no z^{-1} term. Thus,

$$Res[\frac{\sin z}{z^3}, 0] = 0.$$

(e) We have

$$\frac{1+z^2}{z(z-1)^2} = (z^2+1)\cdot\frac{1}{z}\cdot\frac{1}{(z-1)^2}.$$

So $\frac{1+z^2}{z(z-1)^2}$ has a simple pole at the origin and pole of order 2 at $z = 1$.

$$Res[\frac{1+z^2}{z(z-1)^2}, 1] = ((z^2+1)\cdot\frac{1}{(z-1)^2})|_{z=0} = 1,$$

$$Res[\frac{1+z^2}{z(z-1)^2}, 0] = \frac{1}{(2-1)!}\frac{d((z-1)^2\frac{1+z^2}{z(z-1)^2})}{dz}|_{z=1} = \frac{d(\frac{1+z^2}{z})}{dz}|_{z=1}$$

$$= \left(2-\frac{1+z^2}{z^2}\right)\Bigg|_{z=1} = 0.$$

(f) We have

$$\frac{1}{(z^2+a^2)^2} = \frac{1}{(z+ai)^2}\cdot\frac{1}{(z-ai)^2}.$$

So $\frac{1}{(z^2+a^2)^2}$ has two poles of order 2 (double poles). One pole is at $z = ai$ and the other one is at $z = -ai$.

$$Res[\frac{1}{(z^2+a^2)^2}, ai] = \frac{d(z+ai)^{-2}}{dz}|_{z=ai} = \frac{1}{4a^3i},$$

$$Res[\frac{1}{(z^2+a^2)^2}, -ai] = \frac{d(z-ai)^{-2}}{dz}|_{z=-ai} = -\frac{1}{4a^3i}.$$

(g) We have

$$\frac{e^{az}}{2z^2-5z+2} = \frac{e^{az}}{2(z-2)(z-\frac{1}{2})}.$$

Thus, $\frac{e^{az}}{2z^2-5z+2}$ has simple poles at $z = 2$ and $z = \frac{1}{2}$.

$$Res[\frac{e^{az}}{2z^2-5z+2}, 2] = \frac{e^{az}}{2(z-\frac{1}{2})}|_{z=2} = \frac{e^{2a}}{3},$$

$$Res[\frac{e^{az}}{2z^2-5z+2}, \frac{1}{2}] = \frac{e^{az}}{2(z-2)}|_{z=\frac{1}{2}} = -\frac{e^{\frac{a}{2}}}{3}.$$

(h) We have

$$\frac{e^{z-1}-1}{1-z^2} = \frac{-1+\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}}{(1-z)(1+z)} = \frac{1}{1+z}\cdot[\sum_{n=0}^{\infty}\frac{(z-1)^n}{(n+1)!}].$$

Note that $\sum_{n=0}^{\infty} \frac{(z-1)^n}{(n+1)!}$ is analytic for every (finite) z. Thus, the given function is analytic at $z = 1$, with residue equal to 0 at $z = 1$. Since

$$\sum_{n=0}^{\infty} \frac{(z-1)^n}{(n+1)!}|_{z=-1} = \frac{e^{z-1}-1}{1-z}|_{z=-1} = \frac{e^{-2}-1}{2} \neq 0,$$

it's clear that $\frac{e^{z-1}-1}{1-z^2}$ has a simple pole at $z = -1$, and

$$Res[\frac{e^{z-1}-1}{1-z^2}, -1] = \frac{e^{-2}-1}{2}.$$

(i) Evidently, the only pole of $\frac{1-\cos az}{z^9}$ is at $z = 0$. We have

$$\frac{1-\cos az}{z^9} = z^{-9}(1 - \sum_{n=0}^{\infty} \frac{(-1)^n (az)^{2n}}{(2n)!}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^{2n} z^{2n-9}}{(2n)!}.$$

Specially, the coefficient of z^{-1} is $\frac{-a^8}{8!}$. So

$$Res[\frac{1-\cos az}{z^9}, 0] = \frac{-a^8}{8!}.$$

(j) It's easy to check that $\cosh z - 1 = 0$ for $z = i2n\pi$ with n: integer. Thus, the only possible singularities are poles at $z = z_n = i2n\pi$. In order to examine what sort of poles these are, let $t = z - z_n$. Then,

$$\frac{\sinh z}{\cosh z - 1} = \frac{\sinh t}{\cosh t - 1} = \frac{\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}}{-1 + \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}} = t^{-1} \frac{\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+1)!}}{\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+2)!}},$$

and

$$\frac{\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+1)!}}{\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+2)!}}|_{t=0} = 2 \neq 0.$$

So $t = 0$, i.e., $z = z_n = i2n\pi$, is a simple pole of $\frac{\sinh z}{\cosh z - 1}$, and

$$Res[\frac{\sinh z}{\cosh z - 1}, 0] = 2.$$

(k) $\sin z$ has zeros at $z = k\pi$, where k: integer, and each of these zeros is simple. Accordingly, $\frac{z}{\sin^2 z}$ has a simple pole at $z = 0$ and poles of order 2 at $z = k\pi$, where $k \neq 0$.

$$Res[\frac{z}{\sin^2 z}, 0] = \lim_{z \to 0} z \frac{z}{\sin^2 z} = 1,$$

and, for $k \neq 0$ (k: integer),

$$Res[\frac{z}{\sin^2 z}, k\pi] = \lim_{z \to k\pi} \frac{d}{dz}((z-k\pi)^2 \frac{z}{\sin^2 z}) = 1.$$

Alternatively, by setting $t = z - k\pi$,

$$\frac{z}{\sin^2 z} = \frac{t+k\pi}{\sin^2 t} \sim \frac{t+k\pi}{t^2} \sim \frac{k\pi}{t^2} + \frac{1}{t}.$$

For $k = 0$, the t^{-2} term vanishes and, hence, $z = 0$ is a simple pole. For $k \neq 0$, the t^{-2} term is nonzero and, hence, $z = k\pi, k \neq 0$, is a double pole. In each case, the coefficient of the t^{-1} term is 1. Thus, the residue is 1.

(l) The only possible pole of $\frac{(1-\cos z)^2}{z^7}$ is $z = 0$. Let

$$g(z) = \frac{\cos z - 1}{z^2}.$$

Then $g(z)$ is analytic, $g(0) = -\frac{1}{2} \neq 0$, $g'(0) = 0$, and $g''(0) = \frac{1}{12}$. We have

$$\frac{(1-\cos z)^2}{z^7} = z^{-3}(g(z))^2.$$

So $\frac{(1-\cos z)^2}{z^7}$ has a pole of order 3 at $z = 0$, and

$$\begin{aligned} Res[\frac{(1-\cos z)^2}{z^7}, 0] &= \frac{1}{(3-1)!}\frac{d^2(g(z))^2}{dz^2}|_{z=0} \\ &= \frac{1}{2}\frac{d(2g(z)g'(z))}{dz}|_{z=0} \\ &= ((g'(0))^2 + g(0)g''(0)) \\ &= \frac{-1}{2}\frac{1}{12} = \frac{-1}{24}. \end{aligned}$$

Alternatively, we expand this function in Laurent series as follows:

$$\frac{(1-\cos z)^2}{z^7} = \frac{1}{z^3}\left(\frac{\cos z - 1}{z^2}\right)^2$$

$$= z^{-3}\left(-\frac{1}{2} + \frac{1}{24}z^2 + \dots\right)^2 = z^{-3}\left(\frac{1}{4} - \frac{1}{24}z^2 + \dots\right)$$

$$= \frac{1}{4}\frac{1}{z^3} - \frac{1}{24}\frac{1}{z} + \dots$$

Clearly, the residue is $-1/24$. I personally find this alternative way faster!

Note: In the above, we use the symbol $\sim$ to mean “approximately equal to” in cases where we neglect the other terms in Laurent series.

79. If $f(z)$ has a pole of order m at $z = a$, prove that

$$\text{Res}(a) = \frac{1}{(M-1)!}\left[\frac{d^{M-1}}{dz^{M-1}}\{(z-a)^M f(z)\}\right]_{z=a}$$

for any positive integer M such that $M \geq m$.

Solution. By definition of the point $z = a$ as a pole of order m, $f(z)$ admits the Laurent expansion

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \dots,$$

where c_{-1} is the residue. It follows that

$$(z-z_0)^M\, f(z) = c_{-m}(z-z_0)^{M-m}+c_{-m+1}(z-z_0)^{M-m+1}+\ldots+c_{-1}(z-z_0)^{M-1}+c_0(z-z_0)^M+\ldots$$

We see that, for $M \geq m$, this last expansion is a Taylor series. In particular, the coefficient c_{-1} multiplies $(z-z_0)^{M-1}$ and thus must be equal to the derivative of order $M-1$ of $(z-z_0)^M f(z)$ at $z=a$ divided by $(M-1)!$:

$$c_{-1} = \frac{1}{(M-1)!}\left[\frac{d^{M-1}}{dz^{M-1}}\{(z-a)^M\, f(z)\}\right]_{z=a}.$$

80. (a) If $f(z)$ is that branch of $\log z$ for which $0 \leq \theta_P < 2\pi$, determine the sum of the residues of $f(z)/(z^2+1)$ at its poles.

(b) Proceed as in part (a) when the restriction on θ_P is $-\pi < \theta_P \leq \pi$.

Solution. The denominator in $f(z)/(z^2+1)$ vanishes at $z=\pm i$. Thus, the possible poles are $z=\pm i$. Because the function $(z\mp i)\frac{f(z)}{z^2+1} = \frac{f(z)}{z\pm i}$ is analytic in a vicinity of $z=\pm i$ in the respective branch of $\log z$, and its value at $z=\pm i$ is nonzero, these poles are simple. The value of $\log z$ at $z=re^{i\theta}$ is $\log z = \log r + i(\theta+2k\pi)$ where k: integer and $\log r$ $(r>0)$ is the usual logarithm for real functions. The principal value is found by setting $k=0$ and $\theta=\theta_P$.

(a) At $z=i=e^{i\pi/2}, \theta_P=\pi/2$ while at $z=-i=e^{-i\pi/2}, \theta_P = -i\pi/2+2\pi = 3\pi/2$. Hence, $\log i = i\pi/2$ and $\log(-i) = i3\pi/2$. It follows that

$$Res[\frac{f(z)}{z^2+1}, z=i] = \lim_{z\to i}[(z-i)\,\frac{f(z)}{z^2+1}] = \frac{\log i}{2i} = \frac{\pi}{4},$$

$$Res[\frac{f(z)}{z^2+1}, z=-i] = \lim_{z\to i}[(z+i)\,\frac{f(z)}{z^2+1}] = \frac{\log(-i)}{-2i} = -\frac{3\pi}{4}.$$

So, the desired sum is

$$Res[f(z), z=i] + Res[f(z), z=-i] = -\frac{\pi}{2}.$$

(b) In this case, $z=i$ has $\theta_P=\pi/2$ and $z=-i$ has $\theta_P=-\pi/2$. Hence, $\log i = i\pi/2$ and $\log(-i) = -i\pi/2$. Accordingly,

$$Res[\frac{f(z)}{z^2+1}, z=i] = \lim_{z\to i}[(z-i)\frac{f(z)}{z^2+1}] = \frac{\log i}{2i} = \frac{\pi}{4},$$

$$Res[\frac{f(z)}{z^2+1}, z=-i] = \lim_{z\to i}[(z+i)\frac{f(z)}{z^2+1}] = \frac{\log(-i)}{-2i} = \frac{\pi}{4}.$$

So, the desired sum is

$$Res[\frac{f(z)}{z^2+1}, z=i] + Res[\frac{f(z)}{z^2+1}, z=-i] = \frac{\pi}{2}.$$

81. (a) If $f(z)$ is that branch of the function $e^{az^{1/2}}$ for which $z^{1/2} = r^{1/2}\, e^{i\theta_P/2}$ with $0 \leq \theta_P < 2\pi$, determine the sum of the residues of $f(z)/(z^2+1)$ at its poles.

(b) Proceed as in part (a) when $-\pi < \theta_P \le \pi$.

Solution. Similarly to Prob. 80 above, the denominator in $f(z)/(z^2+1)$ vanishes at $z = \pm i$. Thus, the possible poles are $z = \pm i$. Because $(z \mp i)\frac{f(z)}{z^2+1} = \frac{f(z)}{z \pm i}$ is analytic in the vicinity of $z = \pm i$ in the respective branch of $e^{az^{1/2}}$ and its value at $z = \pm i$ is nonzero, these poles are simple.

(a) At $z = i = e^{i\pi/2}$, $\theta_P = \pi/2$ while at $z = -i = e^{-i\pi/2}, \theta_P = -i\pi/2 + 2\pi = 3\pi/2$. Hence, $(i)^{1/2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$ and $(-i)^{1/2} = e^{i3\pi/4} = \frac{-1+i}{\sqrt{2}}$. It follows that

$$Res[\frac{f(z)}{z^2+1}, z = i] = \lim_{z\to i}[(z-i)\,\frac{f(z)}{z^2+1}] = \frac{e^{a(1+i)/\sqrt{2}}}{2i},$$

$$Res[\frac{f(z)}{z^2+1}, z = -i] = \lim_{z\to i}[(z+i)\,\frac{f(z)}{z^2+1}] = \frac{e^{a(-1+i)/\sqrt{2}}}{-2i}.$$

So, the desired sum is

$$Res[\frac{f(z)}{z^2+1}, z = i] + Res[\frac{f(z)}{z^2+1}, z = -i] = -ie^{ia/\sqrt{2}}\ \frac{e^{a/\sqrt{2}} - e^{-a/\sqrt{2}}}{2}$$
$$= -ie^{ia/\sqrt{2}}\ \sinh(a/\sqrt{2}).$$

(b) In this case, $z = i$ has $\theta_P = \pi/2$ and $z = -i$ has $\theta_P = -\pi/2$. Hence, $(i)^{1/2} = e^{i\pi/4}$ and $(-i)^{1/2} = e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$. Accordingly,

$$Res[\frac{f(z)}{z^2+1}, z = i] = \lim_{z\to i}[(z-i)\frac{f(z)}{z^2+1}] = \frac{e^{a(1+i)/\sqrt{2}}}{2i},$$

$$Res[\frac{f(z)}{z^2+1}, z = -i] = \lim_{z\to i}[(z+i)\frac{f(z)}{z^2+1}] = \frac{e^{a(1-i)/\sqrt{2}}}{-2i}.$$

So, the desired sum is

$$Res[\frac{f(z)}{z^2+1}, z = i] + Res[\frac{f(z)}{z^2+1}, z = -i] = e^{a/\sqrt{2}}\ \frac{e^{ia/\sqrt{2}} - e^{-ia/\sqrt{2}}}{2i}$$
$$= e^{a/\sqrt{2}}\ \sin(a/\sqrt{2}).$$