SOLUTION SET V FOR 18.075–FALL 2004

10. Functions of a Complex Variable

10.15. **Indented contours.** .

110. By making use of integration around suitable indented contours in the complex plane, evaluate the following integrals:
(a) $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)}\, dx$ $(a > 0)$,
(b) $\int_{-\infty}^{\infty} \frac{\sin x}{x(\pi^2-x^2)}\, dx$.

Solution. (a) For $R > 1$ and $0 < \epsilon < 1$, define the contour $C = C_1 + C_2 + C_3 + C_4$, where C_1 is the real interval $[-R, -\epsilon]$, C_2 is the upper half of the circle $|z| = \epsilon$ with clockwise orientation, C_3 is the real interval $[\epsilon, R]$, and C_4 is the upper half of the circle $|z| = R$ with counterclockwise orientation.

Then

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)}\, dx \;=\; Im \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3} \frac{e^{iz}}{z(z^2+a^2)}\, dz \equiv Im \mathsf{P} \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+a^2)}\, dx.$$

Moreover, by ***Theorem 2*** of classnotes,

$$\lim_{R\to\infty} \int_{C_4} \frac{e^{iz}}{z(z^2+a^2)}\, dz \;=\; 0,$$

and, by ***Theorem 4*** of classnotes,

$$\lim_{\epsilon\to 0} \int_{C_2} \frac{e^{iz}}{z(z^2+a^2)}\, dz \;=\; -\pi i\; Res[\frac{e^{iz}}{z(z^2+a^2)}, 0] \;=\; -\frac{\pi i}{a^2}.$$

So,

$$\begin{aligned}
\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3} \frac{e^{iz}}{z(z^2+a^2)}\, dz \;&=\; \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \oint_{C} \frac{e^{iz}}{z(z^2+a^2)}\, dz - \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_2+C_4} \frac{e^{iz}}{z(z^2+a^2)}\, dz \\
&=\; 2\pi i\; Res[\frac{e^{iz}}{z(z^2+a^2)}, ia] - (-\frac{\pi i}{a^2}) \\
&=\; \pi i\, \frac{1-e^{-a}}{a^2}.
\end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)}\, dx \;=\; Im \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3} \frac{e^{iz}}{z(z^2+a^2)}\, dz \;=\; \pi\, \frac{1-e^{-a}}{a^2}.$$

Date: October 16, 2002.

(b) For large $R > 0$ and small $\epsilon > 0$, define the contour $C = C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8$, where C_1, C_3, C_5 and C_7 are the real intervals $[-R, -\pi-\epsilon]$, $[-\pi+\epsilon, -\epsilon]$, $[\epsilon, \pi-\epsilon]$ and $[\pi+\epsilon, R]$, C_2 is the upper half of the circle $|z+\pi| = \epsilon$ with clockwise orientation, C_4 is the upper half of the circle $|z| = \epsilon$ with clockwise orientation, C_6 is the upper half of the circle $|z-\pi| = \epsilon$ with clockwise orientation, and C_8 is the upper half of the circle $|z| = R$ with counterclockwise orientation.

Then

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(\pi^2 - x^2)}\, dx \;=\; Im \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3+C_5+C_7} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \equiv Im\, \mathsf{P} \int_{-\infty}^{\infty} \frac{e^{ix}}{x(\pi^2 - x^2)}\, dx.$$

By ***Theorem 2*** of classnotes,

$$\lim_{R\to\infty} \int_{C_8} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \;=\; 0,$$

and, by ***Theorem 4*** of classnotes,

$$\lim_{\epsilon\to 0} \int_{C_2} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \;=\; -\pi i\; Res[\frac{-1}{z(\pi^2 - z^2)}, -\pi] \;=\; \frac{-i}{2\pi},$$

$$\lim_{\epsilon\to 0} \int_{C_4} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \;=\; -\pi i\; Res[\frac{e^{iz}}{z(\pi^2 - z^2)}, 0] \;=\; \frac{-i}{\pi},$$

$$\lim_{\epsilon\to 0} \int_{C_6} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \;=\; -\pi i\; Res[\frac{e^{iz}}{z(\pi^2 - z^2)}, \pi] \;=\; \frac{-i}{2\pi}.$$

Since $\frac{e^{iz}}{z(\pi^2 - z^2)}$ has no singularities in the upper half plane, we get

$$\oint_{C} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \;=\; 0,$$

This gives

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3+C_5+C_7} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \;=\; -\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_2+C_4+C_6+C_8} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \;=\; \frac{2i}{\pi}.$$

So

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(\pi^2 - x^2)}\, dx \;=\; Im \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3+C_5+C_7} \frac{e^{iz}}{z(\pi^2 - z^2)}\, dz \;=\; \frac{2}{\pi}.$$

111. Show that

$$\mathsf{P} \int_{-\infty}^{\infty} \frac{e^{itx}}{x}\, dx \;=\; \begin{cases} \pi i & (t > 0), \\ 0 & (t = 0), \\ -\pi i & (t < 0), \end{cases}$$

and hence also that

$$\mathsf{P} \int_{-\infty}^{\infty} \frac{\cos tx}{x}\, dx \;=\; 0,$$

and

$$\int_{-\infty}^{\infty} \frac{\sin tx}{x}\, dx \;=\; \begin{cases} \pi & (t > 0), \\ 0 & (t = 0), \\ -\pi & (t < 0). \end{cases}$$

Solution. Case 1, $t > 0$. For large R and small ϵ, define the contour $C = C_1 + C_2 + C_3 + C_4$, where C_1 and C_3 are the real intervals $[-R, -\epsilon]$ and $[\epsilon, R]$, C_2 is the upper half of the circle $|z| = \epsilon$ with clockwise orientation, and C_4 is the upper half of the circle $|z| = R$ with counterclockwise orientation.

Then

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{e^{itx}}{x}\, dx \;=\; \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3} \frac{e^{itz}}{z}\, dz.$$

By ***Theorem 2*** of classnotes,

$$\lim_{R\to\infty} \int_{C_4} \frac{e^{itz}}{z}\, dz \;=\; 0,$$

and, by ***Theorem 4*** of classnotes,

$$\lim_{\epsilon\to 0} \int_{C_2} \frac{e^{itz}}{z}\, dz \;=\; -\pi i \; Res[\frac{e^{itz}}{z}, 0] \;=\; -\pi i.$$

Since $\frac{e^{itz}}{z}$ has no singularities on the upper half plane, we get

$$\oint_{C} \frac{e^{itz}}{z}\, dz \;=\; 0.$$

This gives

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{e^{itx}}{x}\, dx \;=\; \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3} \frac{e^{itz}}{z}\, dz \;=\; -\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_2+C_4} \frac{e^{itz}}{z}\, dz \;=\; \pi i.$$

Case 2, $t < 0$. For large R and small ϵ, define the contour $C = C_1 + C_2 + C_3 + C_4$, where C_1 and C_3 are the real intervals $[-R, -\epsilon]$ and $[\epsilon, R]$, C_2 is the lower half of the circle $|z| = \epsilon$ with counterclockwise orientation, and C_4 is the lower half of the circle $|z| = R$ with clockwise orientation.

Then

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{e^{itx}}{x}\, dx \;=\; \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{C_1+C_3} \frac{e^{itz}}{z}\, dz.$$

By ***Theorem 2*** of classnotes,

$$\lim_{R\to\infty} \int_{C_4} \frac{e^{itz}}{z}\, dz \;=\; 0,$$

and, by ***Theorem 4*** of classnotes,

$$\lim_{\epsilon\to 0} \int_{C_2} \frac{e^{itz}}{z}\, dz \;=\; \pi i \; Res[\frac{e^{itz}}{z}, 0] \;=\; \pi i.$$

Since $\frac{e^{itz}}{z}$ has no singularities in the lower half plane, we get

$$\oint_C \frac{e^{itz}}{z}\, dz \;=\; 0.$$

This gives

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{e^{itx}}{x}\, dx \;=\; \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{C_1+C_3} \frac{e^{itz}}{z}\, dz \;=\; -\lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{C_2+C_4} \frac{e^{itz}}{z}\, dz \;=\; -\pi i.$$

Case 3, $t = 0$. Since $\frac{1}{x}$ is odd, it's clear that

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{1}{x}\, dx \;=\; 0.$$

Combine all three cases, we get

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{e^{itx}}{x}\, dx \;=\; \begin{cases} \pi i & (t > 0), \\ 0 & (t = 0), \\ -\pi i & (t < 0), \end{cases}$$

Compare the real and imaginary parts on both sides. We get

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{\cos tx}{x}\, dx \;=\; 0,$$

and

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{\sin tx}{x}\, dx \;=\; \begin{cases} \pi & (t > 0), \\ 0 & (t = 0), \\ -\pi & (t < 0). \end{cases}$$

Note that the integral $\int_{-\infty}^{\infty} \frac{\sin tx}{x}\, dx$ actually converges in all three cases. So the last equation above becomes

$$\int_{-\infty}^{\infty} \frac{\sin tx}{x}\, dx \;=\; \begin{cases} \pi & (t > 0), \\ 0 & (t = 0), \\ -\pi & (t < 0). \end{cases}$$

116. Obtain the evaluation

$$\int_{-\infty}^{\infty} dx\, \frac{\cos ax - \cos bx}{x^2} = \pi(b - a).$$

[Notice that $f(z) = (e^{iaz} - e^{ibz})/z^2$ has a simple pole at the origin.] By taking $a = 0$ and $b = 2$, also deduce the formula

$$\int_{-\infty}^{\infty} dx\, \frac{\sin^2 x}{x^2} = \pi.$$

Solution. If we naively set $\cos ax = Re(e^{iax})$ and $\cos bx = Re(e^{ibx})$ and take the Re outside the integral sign, then the resulting integral doesn't make any sense as is:

$$I = \int_{-\infty}^{\infty} dx\ \frac{\cos ax - \cos bx}{x^2} \neq Re \int_{-\infty}^{\infty} dx\ \frac{e^{iax} - e^{ibx}}{x^2} = \infty.$$

More precisely, I itself is finite, since the integrand in the left-hand side is well-behaved for all x. For example, by expanding the cosines in the integrand near $z = 0$, we find

$$\begin{aligned} \frac{\cos az - \cos bz}{z^2} &= \frac{[1 - \frac{(az)^2}{2!} + \ldots + \frac{(-1)^n (az)^{2n}}{(2n)!} + \ldots] - [1 - \frac{(bz)^2}{2!} + \ldots + \frac{(-1)^n (bz)^{2n}}{(2n)!} + \ldots]}{z^2} \\ &= -\frac{a^2 - b^2}{2!} + \ldots + (-1)^n \frac{a^{2n} - b^{2n}}{(2n)!} z^{2(n-1)} + \ldots . \end{aligned}$$

On the other hand, the integrand on the right-hand side, $f(z) \equiv \frac{e^{iaz} - e^{ibz}}{z^2}$, has a simple pole at $z = 0$. Indeed $zf(z)$ reads as

$$\begin{aligned} zf(z) &= \frac{e^{iaz} - e^{ibz}}{z} \\ &= \frac{[1 + iaz + \frac{(iaz)^2}{2!} + \ldots + \frac{(iaz)^n}{n!} + \ldots] - [1 + ibz + \frac{(ibz)^2}{2!} + \ldots + \frac{(ibz)^n}{n!} + \ldots]}{z} \\ &= i(a - b) - \frac{a^2 - b^2}{2!} z + \ldots + i^n \frac{a^n - b^n}{n!} z^{n-1} + \ldots , \end{aligned}$$

which is analytic at $z = 0$ (and furthermore $\lim_{z \to 0}[zf(z)] \neq 0$ since $a \neq b$).

Therefore, we read $I = ReI_p$ where I_p is the principal-value integral

$$I_p \equiv \mathsf{P} \int_{-\infty}^{\infty} dx\ \frac{e^{iax} - e^{ibx}}{x^2}.$$

We calculate I_p by closing the path by a small semicircle $C_{\epsilon+}$ of radius ϵ around $z = 0$ in the upper half plane, and a large semicircle C_{R+} of radius R also in the upper half plane. The resulting closed contour does not contain any singularities of the integrand and has to be zero by the Cauchy integral theorem. In addition, by ***Theorems 4& 2*** of classnotes,

$$\lim_{\epsilon \to 0^+} \int_{C_{\epsilon+}} dz\, f(z) = -i\pi\, Rez[f(z), 0] = -i\pi\, i(a - b) = \pi(a - b),$$

$$\lim_{R \to \infty} \int_{C_{R+}} dz\, f(z) = 0.$$

The last equation was obtained by noticing that $a > 0$ and $b > 0$ while $1/z^2$ goes to 0 uniformly in $|z| = R$ as $R \to \infty$. It follows that

$$I_p + \lim_{\epsilon \to 0^+} \int_{C_{\epsilon+}} dz\, f(z) + \lim_{R \to \infty} \int_{C_{R+}} dz\, f(z) = 0$$

$$\Rightarrow\ I_p = \pi(b - a)\ \Rightarrow\ I = ReI_p = \pi(b - a).$$

In the special case $a = 0$ and $b = 2$, the integrand of the original integral becomes

$$\frac{\cos ax - \cos bx}{x^2} = \frac{1 - \cos 2x}{x^2} = \frac{2 \sin^2 x}{x^2}.$$

Hence, the result of integration reads as

$$\int_{-\infty}^{\infty} \frac{2\sin^2 x}{x^2}\, dx = 2\pi,$$

which agrees with the desired formula.