SOLUTION SET VI FOR 18.075–FALL 2004

4. Series Solutions of Differential Equations:Special Functions

4.2. **Illustrative examples.** .

5. Obtain the general solution of each of the following differential equations in terms of Maclaurin series:
(a) $\frac{d^2y}{dx^2} = xy$,
(b) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$.

Solution. (a) Try the Maclaurin series $y = \sum_{n=0}^{\infty} a_n x^n$ to get

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} a_{n-1} x^n, \quad \underline{a_{-1} = 0},$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

The differential equation yields

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}]x^n = 0,$$

which is satisfied by all x in some neighborhood of $x_0 = 0$. Hence, the recurrence formula (relation) for the coefficients a_n reads

$$(n+2)(n+1)a_{n+2} = a_{n-1}; \quad a_{-1} = 0, \quad n = 0,\ 1,\ 2,\ 3,\ \dots\ .$$

Find the coefficients explicitly for various n:

$$n = 0: \quad a_2 = 0$$
$$n = 1: \quad 3 \cdot 2a_3 = a_0$$
$$n = 2: \quad 4 \cdot 3a_4 = a_1$$
$$n = 3: \quad 5 \cdot 4a_5 = a_2$$
$$n = 4: \quad 6 \cdot 5a_6 = a_3$$
$$n = 5: \quad 7 \cdot 6a_7 = a_4$$
$$n = 6: \quad 8 \cdot 7a_8 = a_5,\ \dots\ .$$

Notice that a_0 and a_1 are independent and arbitrary, while all coefficients $a_2, a_5, a_8, \dots\ a_{3n+2}\ \dots =$ 0.

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The corresponding power series for $y(x)$ reads as

$$\begin{aligned} y(x) &= a_1\left(x + \frac{x^4}{4\cdot 3} + \frac{x^7}{(3\cdot 4)(6\cdot 7)} + \ldots + \frac{x^{3n+1}}{(3\cdot 4)(6\cdot 7)\ldots[3n\cdot(3n+1)]} + \ldots\right) \\ &\quad + a_0\left(1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{(2\cdot 3)(5\cdot 6)} + \ldots + \frac{x^{3n}}{(2\cdot 3)(5\cdot 6)\ldots[(3n+2)(3n+3)]} + \ldots\right). \end{aligned}$$

(b) Once again, we try the Maclaurin series $y(x)\sum_{n=0}^{\infty} a_n x^n$ to get

$$x\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^n, \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n,$$

which in turn lead to the equation

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + (n-1)a_n]x^n = 0,$$

satisfied by all x in some neighborhood of $x_0 = 0$. It follows that

$$(n+2)(n+1)a_{n+2} = -(n-1)a_n, \quad n = 0,\, 1,\, 2,\, 3,\, \ldots .$$

Write the ensuing coefficients explicitly:

$$\begin{aligned} n = 0: &\quad 2a_2 = a_0, \\ n = 1: &\quad 3\cdot 2a_3 = 0\cdot a_1 = 0, \\ n = 2: &\quad 4\cdot 3a_4 = -a_2, \\ n = 3: &\quad 5\cdot 4a_5 = -2a_3 = 0, \\ n = 4: &\quad 6\cdot 5a_6 = -3a_4, \\ n = 5: &\quad 7\cdot 6a_7 = -4a_5 = 0. \end{aligned}$$

It follows that a_0 and a_1 are independent and arbitrary. Further, all coefficients with odd index are zero, with the exception of a_1 (since the right-hand side of the equation for $n = 1$ vanishes).

The final Maclaurin series for $y(x)$ reads as

$$\begin{aligned} y(x) &= a_0\left(1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{1\cdot 3x^6}{6!} - \frac{1\cdot 3\cdot 5x^8}{8!} + \ldots \right. \\ &\quad \left. + (-1)^n\frac{1\cdot 3\cdot\ \ldots\ (2n-1)x^{2n+2}}{(2n)!} + \ldots\right) + a_1 x. \end{aligned}$$

Notice that the independent solution involving a_1 is $u(x) = x$.

6. For each of the following equations, obtain the most general solution which is representable by a Maclaurin series:

(a) $\frac{d^2y}{dx^2} + y = 0$,

(b) $\frac{d^2y}{dx^2} - (x-3)y = 0$,

(c) $\left(1 - \frac{1}{2}x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$,

(d) $x^2\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$,

(e) $(x^2+x)\frac{d^2y}{dx^2} - (x^2-2)\frac{dy}{dx} - (x+2)y = 0$.
Obtain three nonvanishing terms in each infinite series involved.

Solution. (a) With $y(x) = \sum_{n=0}^{\infty} A_n x^n$, the recurrence formula for the coefficients A_n is

$$(n+2)(n+1)A_{n+2} + A_n = 0, \qquad n = 0, 1, 2, 3, \dots .$$

Specifically,

$$\begin{aligned}
n=0: &\quad 2\cdot 1 A_2 + A_0 = 0 \Rightarrow A_2 = -\frac{A_0}{2\cdot 1},\\
n=1: &\quad 3\cdot 2 A_3 + A_1 = 0 \Rightarrow A_3 = -\frac{A_1}{2\cdot 3},\\
n=2: &\quad 4\cdot 3 A_4 + A_2 = 0 \Rightarrow A_4 = -\frac{A_2}{3\cdot 4} = \frac{A_0}{4!},\\
n=3: &\quad 5\cdot 4 A_5 + A_3 = 0 \Rightarrow A_5 = -\frac{A_3}{5\cdot 4} = \frac{A_1}{5!}, \dots
\end{aligned}$$

It follows that

$$y(x) = A_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + A_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right).$$

(b) Again, start with $y(x) = \sum_{n=0}^{\infty} A_n x^n$ and $xy(x) = \sum_{n=0}^{\infty} A_{n-1} x^n$, where $\underline{A_{-1} = 0}$, to arrive at the recurrence formula

$$(n+2)(n+1)A_{n+2} - A_{n-1} + 3A_n = 0; \quad A_{-1} = 0, \quad n = 0, 1, 2, \dots .$$

Specifically,

$$\begin{aligned}
n=0: &\qquad 2\cdot 1 A_2 + 3A_0 = 0 \Rightarrow A_2 = -\frac{3}{1\cdot 2}A_0,\\
n=1: &\qquad 3\cdot 2 A_3 - A_0 + 3A_1 = 0 \Rightarrow A_3 = \frac{A_0}{2\cdot 3} - \frac{A_1}{2},\\
n=2: &\qquad 4\cdot 3 A_4 - A_1 + 3A_2 = 0 \Rightarrow A_4 = \frac{A_1}{3\cdot 4} - \frac{A_2}{4} = \frac{A_1}{3\cdot 4} + \frac{3A_0}{8}, \dots .
\end{aligned}$$

It follows that

$$\begin{aligned}
y(x) &= A_0 + A_1 x - \frac{3}{2}A_0 x^2 + \left(\frac{A_0}{6} - \frac{A_1}{2}\right)x^3 + \left(\frac{A_1}{12} + \frac{3A_0}{8}\right)x^4 + \dots\\
&= A_0\left(1 - \frac{3}{2}x^2 + \frac{1}{6}x^3 - \dots\right) + A_1\left(x - \frac{x^3}{2} + \frac{x^4}{12} - \dots\right).
\end{aligned}$$

(c) With $y(x) = \sum_{n=0}^{\infty} A_n x^n$, we get

$$x\frac{dy}{dx} = \sum_{n=0}^{\infty} nA_n x^n, \quad x^2\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)A_n x^n,$$

and we find the recurrence formula

$$(n+2)(n+1)A_{n+2} - \frac{1}{2}(n-1)(n-2)A_n = 0.$$

Try different values of n:

$$\begin{aligned} n=0: &\qquad 2\cdot 1A_2 - A_0 = 0 \Rightarrow A_2 = \frac{A_0}{2}, \\ n=1: &\qquad 3\cdot 2A_3 - 0 = 0 \Rightarrow A_3 = 0, \\ n=2: &\qquad 4\cdot 3A_4 = 0, \\ n=3: &\qquad 5\cdot 4A_5 = A_3 = 0, \\ n=4: &\qquad 6\cdot 5A_6 - 3A_4 = 0 \Rightarrow A_6 = 0, \\ n=5: &\qquad 7\cdot 6A_7 - 2\cdot 3A_5 = 0 \Rightarrow A_7 = 0 \quad \textit{etc.} \end{aligned}$$

It follows that all coefficients A_n with $n \geq 3$ vanish! Hence,

$$y(x) = A_0\left(1 + \frac{x^2}{2}\right) + A_1 x.$$

(d) Clearly,

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+1)A_{n+1}x^n,$$

$$x^2\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)A_n x^n.$$

The recurrence formula is

$$[n(n-1)+1]A_n = (n+1)A_{n+1}, \quad n = 0,\, 1,\, 2,\, \dots ..$$

Specifically,

$$n=0: \quad A_0 = A_1,$$

$$n=1: \quad A_1 = 2A_2 \Rightarrow A_2 = \frac{A_0}{2},$$

$$n=2: \quad 3A_2 = 3A_3 \Rightarrow A_3 = \frac{A_0}{2} \quad \textit{etc.}$$

Hence,

$$y(x) = A_0\left(1 + x + \frac{x^2}{2} + \dots\right).$$

(e) Clearly,

$$(x+2)y = \sum_{n=0}^{\infty} A_{n-1}x^n + 2\sum_{n=0}^{\infty} A_n x^n, \quad \underline{A_{-1} = 0},$$

$$(x^2-2)\frac{dy}{dx} = \sum_{n=0}^{\infty}(n-1)A_{n-1}x^n - 2\sum_{n=0}^{\infty}(n+1)A_{n+1}x^n,$$

$$(x^2+x)\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)A_n x^n + \sum_{n=0}^{\infty} n(n+1)A_{n+1}x^n.$$

By putting all these terms together, the recurrence formula reads

$$(n-2)(n+1)A_n + (n+1)(n+2)A_{n+1} - nA_{n-1} = 0; \quad \underline{A_{-1} = 0}, \quad n = 0,\, 1,\, 2,\, \dots .$$

Specifically,

$$n = 0: \quad -2A_0 + 1 \cdot 2A_1 = 0 \Rightarrow A_0 = A_1,$$

$$n = 1: \quad -2A_1 + 2 \cdot 3A_2 - A_0 = 0 \Rightarrow A_2 = \frac{A_0}{2} \quad etc.$$

Finally,

$$y(x) = A_0\left(1 + x + \frac{x^2}{2} + \dots\right).$$

4.3. **Singular points of linear, second-order differential equations.** .

8. Locate and classify the singular points of the following differential equations:
(a) $(x-1)y'' + \sqrt{x}y = 0$ $(x \geq 0)$,
(b) $y'' + y' \log x + xy = 0$ $(x \geq 0)$,
(c) $xy'' + y \sin x = 0$,
(d) $y'' - |1 - x^2|y = 0$,
(e) $y'' + y \cos \sqrt{x} = 0$ $(x \geq 0)$.

Solution. (a) The singular points are $x = 1$ and $x = 0$. $x = 1$ is a regular singular point since $(x-1)^2 \cdot \frac{\sqrt{x}}{(x-1)} = (x-1)\sqrt{x}$ has a Taylor expansion near $x = 1$. Since $(x^2 \cdot \frac{\sqrt{x}}{(x-1)})'''|_{x=0}$ does not exist, $x^2 \cdot \frac{\sqrt{x}}{(x-1)}$ does not have a Taylor expansion near $x = 0$. So $x = 0$ is a irregular singular point.

(b) The singular point is $x = 0$, which is irregular since $x \log x$ is not differentiable at $x = 0$.

(c) There are no singular points. (Note that $\frac{\sin x}{x} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-2}}{(2n-1)!}$.)

(d) The singular points are $x = 1$ and $x = -1$. Since neither $((x-1)^2 \cdot |1-x^2|)''|_{x=1}$ nor $((x+1)^2 \cdot |1-x^2|)''|_{x=-1}$ is well defined, both singular points are irregular.

(e) There are no singular points. (Note that $\cos\sqrt{x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(2n)!}$.)

4.4. **The Method of Frobenius.** .

11. Use the method of Frobenius to obtain the general solution of each of the following differential equations, valid near $x = 0$:
(a) $2xy'' + (1-2x)y' - y = 0$,
(b) $x^2y'' + xy + (x^2 - \frac{1}{4})y = 0$,
(c) $xy'' + 2y' + xy = 0$,
(d) $x(1-x)y'' - 2y' + 2y = 0$.

Solution. (a)Rewrite the equation as

$$y'' + \frac{1}{x}(\frac{1}{2} - x)y' + \frac{1}{x^2}(-\frac{x}{2})y = 0.$$

Then we can see that $P_0 = 1/2$, $P_1 = -1$,$Q_1 = -1/2$,and all other P_n's, Q_n's and R_n's are zeros. So $f(s) = s^2 - \frac{1}{2}s$, $g_1(s) = -s + 1/2$, and $g_n(s) = 0$ if $n \neq 1$. $f(s) = 0$ has two roots: $s = \frac{1}{2}$ and $s = 0$. Take $s = 0$, then $A_n = \frac{A_{n-1}}{n}$, for all $n \geq 1$. Hence, by induction, $A_n = \frac{A_0}{n!}$ for all $n \geq 0$. Therefore

$$y = A_0 \sum_{n=1}^{\infty} \frac{x^n}{n!} = A_0 e^x$$

Now, take $s = 1/2$, then $A_n = 2\frac{A_{n-1}}{2n+1}$, for all $n \geq 1$. Therefore

$$\begin{aligned} y &= x^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n x^n \\ &= x^{\frac{1}{2}} A_0 \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^n. \end{aligned}$$

Here $(2n + 1)!! = 3 \cdot 5 \cdot 7 ... \cdot (2n + 1.)$

The general solution is then of the form:

$$y(x) = C_1 e^x + C_2 x^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^n \right).$$

(b) Rewrite the equation as

$$y'' + \frac{1}{x} y' + \frac{1}{x^2}(x^2 - \frac{1}{4})y \;=\; 0.$$

Then we can see that $P_0 = 1$, $Q_0 = -\frac{1}{4}$, $Q_2 = 1$, and all other P_n's, Q_n's and R_n's are zeros. So $f(s) = s^2 - \frac{1}{4}$, $g_2(s) = 1$, and $g_n(s) = 0$ if $n \neq 2$. $f(s) = 0$ has two roots: $s = \frac{1}{2}$ and $s = -\frac{1}{2}$.

For $s = -\frac{1}{2}$ we have $A_n = -\frac{1}{n(n-1)} A_{n-2}$ for all $n \geq 2$. From this, it easy to check by induction that $A_{2n} = \frac{(-1)^n}{(2n)!} A_0$ and $A_{2n+1} = \frac{(-1)^n}{(2n+1)!} A_1$ for all $n \geq 0$. So, in this case,

$$\begin{aligned} y &= x^{-\frac{1}{2}} \sum_{n=0}^{\infty} A_n x^n \\ &= A_0 x^{-\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right) + A_1 x^{-\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right) \\ &= A_0 x^{-\frac{1}{2}} \cos x + A_1 x^{-\frac{1}{2}} \sin x. \end{aligned}$$

The general solution is then of the form

$$y \;=\; c_0 x^{-\frac{1}{2}} \cos x + c_1 x^{-\frac{1}{2}} \sin x.$$

(c) Rewrite the equation as

$$y'' + \frac{2}{x} y' + \frac{x^2}{x^2} y \;=\; 0.$$

The we can see that $P_0 = 2$, $Q_2 = 1$, and all other P_n's, Q_n's and R_n's are zeros. So $f(s) = s^2 + s$, $g_2(s) = 1$, and $g_n(s) = 0$ if $n \neq 2$. $f(s) = 0$ has two roots: $s = -1$ and $s = 0$.

For $s=-1$, we have $A_n = -\frac{1}{n(n-1)}A_{n-2}$. So $A_{2n} = \frac{(-1)^n}{(2n)!}A_0$ and $A_{2n+1} = \frac{(-1)^n}{(2n+1)!}A_1$ for all $n \geq 0$. Then

$$\begin{aligned} y &= x^{-1}\sum_{n=0}^{\infty} A_n x^n \\ &= x^{-1}(A_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} + A_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}) \\ &= x^{-1}(A_1 \sin x + A_0 \cos x). \end{aligned}$$

The general solution is then of the form

$$y = x^{-1}(c_1 \sin x + c_0 \cos x).$$

(d) Rewrite the equation as

$$(1-x)y'' - \frac{2}{x}y' + \frac{2x}{x^2}y = 0.$$

Then we can see $R_1 = -1$, $P_0 = -2$, $Q_1 = 2$, and all other P_n's, Q_n's and R_n's are zeros. So $f(s) = s^2 - 3s$, $g_1(s) = -s^2 + 3s$, and $g_n(s) = 0$ for all $n > 1$. $f(s)$ has two roots: $s = 3$ and $s = 0$.

For $s = 0$, $A_n = -\frac{g_1(n)}{f(n)}A_{n-1} = A_{n-1}$ for all $n \geq 1$, $n \neq 3$. Thus, $A_2 = A_1 = A_0$, and $A_3 = A_4 = A_5 = \cdots$. So, in this case,

$$\begin{aligned} y &= x^0 \sum_{n=0}^{\infty} A_n x^n \\ &= A_0\ (1 + x + x^2) + A_3 x^3 \sum_{n=0}^{\infty} x^n \\ &= A_0 \frac{1-x^3}{1-x} + A_3 \frac{x^3}{1-x}. \end{aligned}$$

The general solution is then of the form

$$y = c_0 \frac{1}{1-x} + c_1 \frac{x^3}{1-x}.$$

12. Use the method of Frobenius to obtain the general solution of each of the following differential equations, valid near $x = 0$:

(a) $x^2y'' - 2xy' + (2 - x^2)y = 0$,
(b) $(x-1)y'' - xy' + y = 0$,
(c) $xy'' - y' + 4x^3y = 0$,
(d) $(1 - \cos x)y'' - \sin xy' + y = 0$.

Solution. (a)Rewrite the equation as

$$y'' - \frac{2}{x}y' + \frac{1}{x^2}(2 - x^2)y = 0.$$

Then we can see that $P_0 = -2$, $Q_0 = 2$,$Q_2 = -1$ and all other P_n's, Q_n's and R_n's are zeros. So $f(s) = s^2 - 3s + 2$, $g_2(s) = -1$, and $g_n(s) = 0$ if $n \neq 2$. $f(s) = 0$ has two roots: $s = 1$ and $s = 2$. For $s = 1$, we have

$$A_n \;=\; \frac{A_{n-2}}{n(n-1)}$$

for $n \geq 2$. From this, it's easy to check by induction that $A_{2n} = \frac{A_0}{(2n)!}$ and $A_{2n+1} = \frac{A_1}{(2n+1)!}$ for all $n \geq 0$. So

$$y \;=\; x\left(A_0\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n} + A_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}\right) \;=\; x(A_0\cosh(x) + A_1\sinh(x)).$$

The general solution is then of the form

$$y \;=\; c_0 x\cosh(x) + c_1 x\sinh(x).$$

(b) Rewrite the equation as

$$(1-x)y'' + xy' - \frac{x^2}{x^2}y \;=\; 0.$$

Then we can see that $R_1 = -1$, $P_2 = 1$, $Q_2 = -1$, and all other P_n's, Q_n's and R_n's are zeros. So $f(s) = s^2 - s$, $g_1(s) = -(s-1)(s-2)$, $g_2(s) = s - 3$, and $g_n(s) = 0$ if $n \geq 3$. $f(s) = 0$ has two roots: $s = 0$ and $s = 1$.

For $s = 0$, we have

$$A_n \;=\; -\frac{g_1(n)A_{n-1} + g_2(n)A_{n-2}}{f(n)} \;=\; \frac{n-2}{n}A_{n-1} - \frac{n-3}{n(n-1)}A_{n-2}$$

for $n \geq 2$. From this, it's easy to check by induction that $A_n = \frac{A_0}{n!}$ if $n \geq 2$. So

$$y \;=\; A_0\left(1 + \sum_{n=2}^{\infty}\frac{x^n}{n!}\right) + A_1 x \;=\; A_0(e^x - x) + A_1 x \;=\; A_0 e^x + (A_1 - A_0)x.$$

Hence the general solution is of the form

$$y \;=\; c_0 e^x + c_1 x.$$

(c) Rewrite the equation as

$$y'' - \frac{1}{x}y' + \frac{4x^4}{x^2}y = 0.$$

Then we can see that $Q_4 = 4$, $P_0 = -1$, and all other P_n's, Q_n's and R_n's are zeros. So $f(s) = s^2 - 2s$, $g_4(s) = 4$, and $g_n(s) = 0$ if $n \neq 4$. $f(s) = 0$ has two roots: $s = 0$ and $s = 2$.

For $s = 0$, we have $A_1 = A_3 = 0$, and $A_n = -\frac{4}{n(n-2)}A_{n-4}$ for all $n \geq 4$. From these, it's easy to check by induction that $A_{2n+1} = 0$, $A_{4n} = \frac{(-1)^n}{(2n)!}A_0$, and $A_{4n+2} = \frac{(-1)^n}{(2n+1)!}A_2$ for all $n \geq 0$. So

$$y \;=\; A_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n} + A_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{4n+2} \;=\; A_0\cos(x^2) + A_2\sin(x^2).$$

The general solution is then of the form

$$y = c_0 \cos(x^2) + c_1 \sin(x^2).$$

(d) Rewrite the equation as

$$\left(\sum_{n=0}^{\infty} 2\frac{(-1)^n}{(2n+2)!}x^{2n}\right) y'' + \frac{1}{x}\left(\sum_{n=0}^{\infty} 2\frac{(-1)^{n+1}}{(2n+1)!}x^{2n}\right) y' + \frac{2}{x^2}y = 0.$$

Then we can see that $Q_0 = 2$, $P_{2n} = 2\frac{(-1)^{n+1}}{(2n+1)!}$, $R_{2n} = 2\frac{(-1)^n}{(2n+2)!}$ for all $n \geq 0$, and all other P_n's, Q_n's and R_n's are zeros. So $f(s) = (s-1)(s-2)$, and $g_{2n-1}(s) = 0$, $g_{2n}(s) = 2\frac{(-1)^n}{(2n+2)!}(s-2n)(s-4n-3)$ for all $n \geq 1$. $f(s) = 0$ has two roots: $s = 1$ and $s = 2$.

For $s = 1$,using the equation

$$f(s+n)A_n = -\sum_{k=1}^{n} g_k(s+n)A_{n-k},$$

it's easy to check by induction that $A_{2n} = \frac{(-1)^n}{(2n+1)!}A_0$, and $A_{2n+1} = 2\frac{(-1)^n}{(2n+2)!}A_1$ for all $n \geq 0$. So

$$\begin{aligned}
y &= x\sum_{n=0}^{\infty} A_n x^n \\
&= A_0 x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n} + A_1 x \sum_{n=0}^{\infty} 2\frac{(-1)^n}{(2n+2)!}x^{2n+1} \\
&= A_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} + A_1 \sum_{n=0}^{\infty} 2\frac{(-1)^n}{(2n+2)!}x^{2n+2} \\
&= A_0 \sin x + 2A_1(1-\cos x).
\end{aligned}$$

The general solution is then of the form

$$y = c_0 \sin x + c_1(1-\cos x).$$