# SOLUTION SET VII FOR 18.075–FALL 2004

4. SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS:SPECIAL FUNCTIONS

## 4.10. **Differential Equations Satisfied by Bessel Functions.** .

**46.** Obtain the general solution of each of the following equation in terms of Bessel functions or, if possible, in terms of elementary functions:

(a) $xy'' - 3y + xy = 0$,

(b) $xy'' - y + 4x^3 y = 0$,

(c) $x^2 y'' + xy' - (x^2 + \frac{1}{4})y = 0$,

(d) $xy'' + (2x + 1)(y' + y) = 0$,

(e) $xy'' - y' - xy = 0$,

(f) $x^4 y'' + a^2 y = 0$,

(g) $y'' - x^2 y = 0$,

(h) $xy'' + (1 + 2x)y' + y = 0$.

(i) $xy'' + (1 + 4x^2)y' + x(5 + 4x^2)y = 0$.

*Solution.* (a) Rewrite the equation as:

$$x^2 y'' - 3xy + x^2 y = 0.$$

Following the method described in section **4.10** and in class (in coming Friday!), let $A = 2, B = 0, C = 1, s = 1, p = 2$.Then,letting $Y = y/x^2, X = x$ we get that $Y$ satisfies the following differential equation:

$$XY''(X) + XY'(X) + (X^2 - 4)Y(X) = 0$$

hence $Y = Z_2(X)$. Therefore, the general solution of the equation in (a) is given by:

$$y = x^2(c_1 Y_2(x) + c_2 J_2(x)).$$

(b) Rewrite the equation as:

$$x^2 y'' - xy + 4x^4 y = 0.$$

Following the method described in section **4.10** and in class, let $A = 1, B = 0, C = 1, s = 2, p = 1/2$. Then,letting $Y = y/x, X = x^2$ we get that $Y$ satisfies the following differential equation:

$$XY''(X) + XY'(X) + (X^2 - 1/4)Y(X) = 0$$

hence $Y = Z_{\frac{1}{2}}(X)$.Therefore, the general solution of the equation in (b) is given by:

$$y = x(c_1 J_{\frac{1}{2}}(x^2) + c_2 J_{-\frac{1}{2}}(x^2)) = C_1 \cos x^2 + C_2 \sin x^2.$$

(c) By setting $X = ix$, the general solution is found to be $y = c_1 I_{\frac{1}{2}}(x) + c_2 I_{-\frac{1}{2}}(x)$. Since $I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$, and $I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cosh x$, the general solution reads

$$y = \sqrt{\frac{2}{\pi x}}(c_1 \sinh x + c_2 \cosh x).$$

(d) Rewrite the equation as

$$x^2 y'' + x(2x+1)y' + (2x^2 + x)y = 0.$$

Following the method described in **4.10** and in class, let $A = 0, B = C = 1, r = 1, s = 1, p = 0$. Then, letting $Y = ye^x, X = x$, we get that $Y$ satisfies the following differential equation:

$$X^2 Y''(X) + XY(X) + X^2 Y(X) = 0,$$

hence

$$Y = Z_0(X) \Rightarrow Y = c_1 J_0(X) + c_2 Y_0(X).$$

Therefore, the general solution of (d) is

$$y = e^{-x}(c_1 J_0(x) + c_2 Y_0(x)).$$

(e) Rewrite the equation as

$$x^2 y'' - xy' - x^2 y = 0.$$

Similar to (d), one can see that if we let $[A = 1, B = 0, r = 0, C = i, p = 1 \text{ and } s = 1]$

$$Y = \frac{y}{x}, \text{ and } X = ix,$$

then $X, Y$ satisfy

$$X^2 Y''(X) + XY'(X) + (X^2 - 1)Y(X) = 0.$$

So $Y = Z_1(X)$, and, hence, the general solution is

$$y = xZ_1(ix) \Rightarrow y = x(c_1 I_1(x) + c_2 K_1(x)).$$

(f) Rewrite the equation as

$$x^2 y'' + \frac{a^2}{x^2}y = 0.$$

Similar to (d), one can see that if we let $[A = \frac{1}{2}, B = 0, r = 0, C = a, p = \frac{1}{2} \text{ and } s = -1]$

$$Y = \frac{y}{\sqrt{x}}, \text{ and } X = \frac{a}{x},$$

then $X, Y$ satisfy

$$X^2 Y''(X) + XY'(X) + (X^2 - \frac{1}{4})Y(X) = 0.$$

So $Y = Z_{\frac{1}{2}}(X)$, and, hence, the general solution is

$$y = \sqrt{x}Z_{\frac{1}{2}}(\frac{a}{x}) \Rightarrow y = x(c_1 \sin(\frac{a}{x}) + c_2 \cos(\frac{a}{x})).$$

(g) Rewrite the equation as

$$x^2 y'' - x^4 y = 0.$$

Similar to (d), one can see that if we let $[A = frac12, B = 0, r = 0, C = \frac{i}{2}, p = \frac{1}{4}$ and $s = 2]$

$$Y = \frac{y}{\sqrt{x}}, \text{ and } X = \frac{ix^2}{2},$$

then $X, Y$ satisfy

$$X^2 Y''(X) + XY'(X) + (X^2 - \frac{1}{16})Y(X) = 0.$$

So $Y = Z_{\frac{1}{4}}(X)$, and, hence, the general solution is

$$y = \sqrt{x} Z_{\frac{1}{4}}(\frac{i}{2}x^2) \Rightarrow y = \sqrt{x}(c_1 I_{\frac{1}{4}}(\frac{x^2}{2}) + c_2 I_{-\frac{1}{4}}(\frac{x^2}{2})).$$

(h) Rewrite the equation as

$$x^2 y'' + x(1 + 2x)y' + xy = 0.$$

Similar to (d), one can see that if we let $[A = 0, B = 1, r = 1, C = i, p = 0$ and $s = 1]$

$$Y = ye^x, \text{ and } X = ix,$$

then $X, Y$ satisfy

$$X^2 Y''(X) + XY'(X) + X^2 Y(X) = 0.$$

So $Y = Z_0(X)$, and, hence, the general solution is

$$y = e^{-x} Z_0(ix) \Rightarrow y = e^{-x}(c_1 I_0(x) + c_2 K_0(x)).$$

(i) Rewrite the equation as

$$x^2 y'' + x(1 + 4x^2)y' + x^2(5 + 4x^2)y = 0.$$

We notice that if we let $[A = 0, B = 1, r = 2, C = 1, p = 0$ and $s = 1]$

$$Y = e^{x^2}y, \qquad X = x,$$

the differential equation becomes

$$X^2 Y''(X) + XY(X) + X^2 Y(X) = 0,$$

which has solution $Y(X) = Z_0(X)$. Hence, the general solution to the given equation is

$$y(x) = e^{-x^2} Z_0(x) \Rightarrow y(x) = e^{-x^2} (c_1 J_0(x) + c_2 Y_0(x)).$$

**47.** The following two equations each have arisen in several physical investigations. Express the general solution of each equation in terms of Bessel functions and also show that it can be expressed in terms of elementary functions when $m$ is an integer:

(a) $y'' - \alpha^2 y = \frac{m(m+1)}{x^2}y,$

(b) $y'' - \frac{2m}{x}y' - \alpha^2 y = 0.$

*Solution.* (a) Rewrite the equation as

$$x^2 y'' - (\alpha^2 x^2 + m(m+1))y = 0.$$

Use the method we used in problem **46**, we get that, if we set $[A = \frac{1}{2}, B = 0, r = 0, C = i\alpha, p = m + \frac{1}{2}$ and $s = 1]$

$$Y = \frac{y}{\sqrt{x}}, \text{ and } X = i\alpha x,$$

then

$$X^2 Y''(X) + XY'(X) + (X^2 - (m + \frac{1}{2})^2)Y(X) = 0.$$

So $Y = Z_{m+\frac{1}{2}}(X)$, and the general solution is

$$y = \sqrt{x}Z_{m+\frac{1}{2}}(i\alpha x) = \begin{cases} \sqrt{x}(c_1 I_{m+\frac{1}{2}}(\alpha x) + c_2 I_{-(m+\frac{1}{2})}(\alpha x)) & \text{if } m + \frac{1}{2} \text{ is not an integer,} \\ \sqrt{x}(c_1 I_{m+\frac{1}{2}}(\alpha x) + c_2 K_{m+\frac{1}{2}}(\alpha x)) & \text{if } m + \frac{1}{2} \text{ is an integer.} \end{cases}$$

Since $I_{m+\frac{1}{2}}$ can be expressed by elementary functions when $m$ is an integer, it's clear that the general solution can also be expressed by elementary functions when $m$ is an integer.

(b) Rewrite the equation as

$$x^2 y'' - 2mxy' - \alpha^2 x^2 y = 0.$$

Use the method we used in problem **46** and in class, we get that, if we set $[A = m + \frac{1}{2}, B = 0, r = 0, C = i\alpha, p = m + \frac{1}{2}$ and $s = 1]$

$$Y = \frac{y}{x^{m+\frac{1}{2}}}, \text{ and } X = i\alpha x,$$

then

$$X^2 Y''(X) + XY'(X) + (X^2 - (m + \frac{1}{2})^2)Y(X) = 0.$$

So $Y = Z_{m+\frac{1}{2}}(X)$, and the general solution is $y(x) = x^{m+\frac{1}{2}}Z_{m+\frac{1}{2}}(i\alpha x)$, which is

$$y(x) = \begin{cases} x^{m+\frac{1}{2}}(c_1 I_{m+\frac{1}{2}}(\alpha x) + c_2 I_{-(m+\frac{1}{2})}(\alpha x)) & \text{if } m + \frac{1}{2} \text{ is not an integer,} \\ x^{m+\frac{1}{2}}(c_1 I_{m+\frac{1}{2}}(\alpha x) + c_2 K_{m+\frac{1}{2}}(\alpha x)) & \text{if } m + \frac{1}{2} \text{ is an integer.} \end{cases}$$

Since $I_{m+\frac{1}{2}}$ can be expressed by elementary functions when $m$ is an integer, it's clear that the general solution can also be expressed by elementary functions when $m$ is an integer.

**48.** Show that for the differential equations

$$xy'' + 3y' + 4xy = 0$$

the condition $y(0) = 1$ determines a unique solution, and hence that $y'(0)$ can not also be prescribed. Determine this solution.

*Solution.* Rewrite the equation as

$$x^2 y'' + 3xy' + 4x^2 y = 0.$$

Similar to problem **46**, one can see that if we let $[A = -1, B = 0, r = 1, C = 2, p = 1, r = 0$ and $s = 1]$

$$Y = xy, \text{ and } X = 2x,$$

then $X, Y$ satisfy

$$X^2 Y''(X) + XY'(X) + (X^2 - 1)Y(X) = 0.$$

So $Y = Z_1(X)$, and, hence, the general solution is
$$y = x^{-1}Z_1(2x) = x^{-1}(c_1 J_1(2x) + c_2 Y_1(2x)).$$

According to the relations (as $x \to 0$)
$$J_p(x) \sim \frac{2^{-p}}{\Gamma(1+p)} x^p, \quad Y_p(x) \sim -\frac{2^p(p-1)!}{\pi} x^{-p},$$

where $p = 1$, it's easy to see that $y(0) = 1$ implies $c_1 = 1$ and $c_2 = 0$ since the second term blows up at $x = 0$! So the condition $y(0) = 1$ determines a unique solution, which is
$$y = x^{-1}J_1(2x).$$

**49.** Find the most general solution of the equation
$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

for which
$$\lim_{x \to 0} 2\pi xy(x) = P$$

where $P$ is a given constant.

*Solution.* This equation is the Bessel equation with $p = 1$. Clearly,
$$y(x) = C_1 J_1(x) + C_2 Y_1(x)$$

is the general solution of the above equation. From the behavior of $J_1(x)$ as $x \to 0$, we deduce that:
$$\lim_{x \to 0} 2\pi x J_1(x) = 0$$

therefore the condition in the exercise is reduced to
$$\lim_{x \to 0} 2\pi x Y_1(x) = P.$$

From the behavior (as $x \to 0$)
$$Y_p(x) \sim -\frac{2^p(p-1)!}{\pi} x^{-p}$$

we get that:
$$\lim_{x \to 0} x Y_1(x) = -2/\pi.$$

Hence the required limit condition implies
$$C_2 = -P/4.$$

Thus the most general solution of
$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

for which
$$\lim_{x \to 0} 2\pi xy(x) = P$$

is :
$$y(x) = C_1 J_1(x) - \frac{P}{4}Y_1(x).$$

**50.** The differential equation for small deflections of a rotating string is of the form

$$\frac{d}{dx}(T\frac{dy}{dx}) + \rho\omega^2 y = 0.$$

Obtain the general solution of this equation under the following assumptions:

(a) $T = T_0 x^n$, $\rho = \rho_0 x^n$; $T_0 = l^2 \rho_0 \omega^2$.
(b) $T = T_0 x^n$, $\rho = \rho_0$, $n \neq 2$; $T_0 = l^2 \rho_0 \omega^2$.
(c) $T = T_0 x^2$, $\rho = \rho_0$; $T_0 = 4\rho_0 \omega^2$.

*Solution.* (a) Rewrite the equation as

$$x^2 y'' + nxy' + \frac{x^2}{l^2}y = 0.$$

By the method used in problem **46**, we get that, if we set $[A = -\frac{n-1}{2}, B = 0, r = 0, C = \frac{1}{l}, p = \frac{n-1}{2}$ and $s = 1]$

$$Y = x^{\frac{n-1}{2}}y, \text{ and } X = \frac{x}{l},$$

then

$$X^2 Y''(X) + XY'(X) + (X^2 - (\frac{n-1}{2})^2)Y(X) = 0.$$

So $Y = Z_{\frac{n-1}{2}}(X)$, and, hence, the general solution is

$$y = x^{\frac{1-n}{2}} Z_{\frac{n-1}{2}}(\frac{x}{l}).$$

(b) Rewrite the equation as

$$x^2 y'' + nxy' + \frac{1}{l^2 x^{n-2}}y = 0.$$

By the method used in problem **46**, we get that, if we set $[A = -\frac{n-1}{2}, B = 0, r = 0, C = \frac{2}{l(2-n)}, p = \frac{(n-1)^2}{(n-2)^2}$ and $s = \frac{2-n}{2}]$

$$Y = x^{\frac{n-1}{2}}y, \text{ and } X = \frac{2}{l(2-n)}x^{\frac{2-n}{2}},$$

then

$$X^2 Y''(X) + XY'(X) + (X^2 - (\frac{n-1}{n-2})^2)Y(X) = 0.$$

So $Y = Z_{\frac{n-1}{n-2}}(X)$, and, hence, the general solution is

$$y = x^{\frac{1-n}{2}} Z_{\frac{n-1}{n-2}}(\frac{2}{l(2-n)}x^{\frac{2-n}{2}}), \qquad n \neq 2.$$

(c) Rewrite the equation as

$$\frac{d}{dx}(4x^2 y'(x)) + y = 0 \Rightarrow x^2 y'' + 2xy' + \frac{1}{4}y = 0.$$

Let $Y = \sqrt{x}y$, then the equation above becomes $[A = -\frac{1}{2}, B = 0, r = 0,$ and $s = 0]$

$$Y''(X) + \frac{1}{x}Y'(X) = 0.$$

So $Y'(X) = cx^{-1}$, and $Y = \int Y'(X)dx = c_1 \log x + c_2$. Then the general solution is

$$y = x^{-\frac{1}{2}}(c_1 \log x + c_2).$$