18.075 Erratum for Solutions to Review Test 3 for Quiz 3, Prob. V.

(V.) (d) The correct function $g_n(s)$ is given by

$$g_n(s) = R_n(s-n)(s-n-1) + P_n(s-n) + Q_n, \quad n \geq 1.$$

Hence, $g_n(s) = 0$ for $n \neq 2$, while

$$g_2(s) = P_1(s-2) + Q_1 = s-2+1 = s-1.$$

$$\Rightarrow g_2(s+k) = s+k-1$$

So, for $s_2 = 1$: $g_2(s_2+k) = k$

$\boxed{\text{Recursive formula for } s = s_2 = 1:}$

$$k(1+k-\lambda) A_k = -k \cdot A_{k-2}, \quad k \geq 1$$

$$\Rightarrow (1+k-\lambda) A_k = -A_{k-2}, \quad k \geq 1. \quad ; \quad \lambda = m : \text{integer}.$$

(e) $\lambda = m > 1$ and $m = 2\ell$: even integer; $\ell = 1, 2, \ldots$

The difference of the 2 roots is $s_1 - s_2 = m - 1 = 2\ell - 1$: odd.

We need to check the recursive formula for $k = s_1 - s_2 = m - 1$ in order to decide whether the Frobenius method gives 2 independent solutions or none.

Suppose that $m = \lambda = 2$, i.e., $\ell = 1$.

$k = m - 1 = 1$: $0 \cdot A_1 = 0 \Rightarrow A_1$: arbitrary ($A_0$: also arbitrary)

Hence, for $\lambda = 2$ the method of Frobenius gives 2 independent solutions

We can continue this way for $\ell = 2, 3, \ldots$, i.e., $\lambda = 4, 6, \ldots$

Take $\ell \geq 2$:

$\underline{k = \overset{m}{\widetilde{2\ell}} - 1}$: $\quad 0 \cdot A_{m-1} = - A_{m-3} \iff 0 \cdot A_{\underbrace{2\ell-1}_{\text{odd}}} = - A_{\underbrace{2\ell-3}_{\text{odd}}}$

What $A_{2\ell-1}$ is depends on what $A_{2\ell-3}$ is. Let's check the recursive formula for lower values of k, $k = 1, 2, \ldots, 2\ell-2$:

$k=1$: $\quad \underbrace{(2-m)}_{\neq 0} A_1 = 0 \implies A_1 = 0$

$\underline{k=2}$: $\quad \underbrace{(3-m)}_{\neq 0} A_2 = - A_0 \implies A_2 = - \dfrac{A_0}{3-m}$

$\underline{k=3}$: $\quad (4-m) A_3 = - A_1 = 0 \implies A_3 = 0$ if $m \neq 4$ etc.

In this way, we can show that $A_1 = A_3 = \cdots = A_{2\ell-3} = 0$, i.e.,

$A_k = 0$ for $k = 1, 3, 5, \ldots, 2\ell-3$.

Hence, for $k = 2\ell-1$ the recursive formula gives

$$0 \cdot A_{2\ell-1} = 0 \implies A_{2\ell-1} : \text{arbitrary} \quad (A_0 : \text{also arbitrary})$$

So, the Frobenius method gives <u>2 independent solutions</u> if <u>λ = even integer.</u>