18.075 Solutions to Practice Test I for Quiz 1, Fall 2004

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(I.) It is sufficient to show that

$$\left.\begin{array}{l} |z_1+z_2|^2 \leq (|z_1|+|z_2|)^2 = |z_1|^2+|z_2|^2+2|z_1 z_2| \\ \text{where } |z_1+z_2|^2 = (z_1+z_2)\cdot(\bar{z}_1+\bar{z}_2) = |z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\bar{z}_2) \end{array}\right\}$$

$$\Leftrightarrow \quad \mathrm{Re}(z_1\bar{z}_2) \leq |z_1\bar{z}_2|.$$

Let $w = z_1\bar{z}_2 = u+iv$. The last inequality is equivalent to

$$u \leq \sqrt{u^2+v^2}.$$

This is true for all u: if $u<0$, it is obviously true.

If $u \geq 0$, the last condition is equivalent (by squaring) to

$$u^2 \leq u^2+v^2 \Leftrightarrow v^2 \geq 0.$$

(II) Let $z = 1-\sqrt{3}\,i$. We need to find $z^{1/3}$.

We find θ_p for z.

$\tan\theta_p = -\sqrt{3}$, and z lies in the 4th quadrant.

$\Rightarrow \theta_p = -\pi/3$ if we take $-\pi < \theta_p \leq \pi$.

The magnitude of z is $|z| = \sqrt{1+3} = 2$.

Thus,

$$z^{1/3} = \left(2\cdot e^{i\theta_p + i2k\pi}\right)^{1/3} = \underbrace{\sqrt[3]{2}}_{>0}\cdot e^{i\frac{\theta_p}{3}+i\frac{2k\pi}{3}}, \quad k=0,1,2.$$

$$= \sqrt[3]{2}\, e^{-i\pi/9}\, e^{i\frac{2k\pi}{3}}$$

III. **1.** $v = 4xy + y$

We check whether v can satisfy the Laplace equation, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

$\frac{\partial^2 v}{\partial x^2} = 0$, $\frac{\partial^2 v}{\partial y^2} = 0$ $\Rightarrow$ $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, so v can be the imaginary part of an analytic function.

2. Suppose $u + iv$ is analytic. Then u and v satisfy the Cauchy-Riemann equations

① $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ② $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

① $\Rightarrow$ $\frac{\partial u}{\partial x} = 4x+1$ $\Rightarrow$ $u(x,y) = 4\frac{x^2}{2} + x + C(y) = 2x^2 + x + C(y)$ ①'

② $\Rightarrow$ $\frac{\partial u}{\partial y} = -4y$ $\overset{①'}{\Rightarrow}$ $C'(y) = -4y$ $\Rightarrow$ $C(y) = -2y^2 + K$.

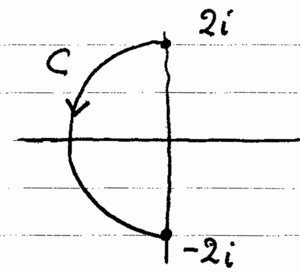
So, $u(x,y) = 2x^2 + x - 2y^2 + K$, K = real const.

3. $f(z) = u + iv = 2x^2 + x - 2y^2 + K + i(4xy + y)$

$= (2x^2 - 2y^2 + i4xy) + x + iy + K$

$= 2(x+iy)^2 + x + iy + K = 2z^2 + z + K$, K: real const.

(IV) $I = \int_C \frac{z^3-2}{z^4}\, dz$



Since it is not specified in the problem statement, we take C to be described in the counterclockwise (positive) sense.

Along C, $z = 2e^{i\theta}$, $\frac{\pi}{2} < \theta < \pi/2 + \pi = \frac{3\pi}{2}$

$\Rightarrow \quad dz = 2ie^{i\theta}\, d\theta$

$$I = \int_C \frac{dz}{z} - 2\int_C \frac{dz}{z^4} = \int_{\pi/2}^{3\pi/2} \frac{\cancel{2}i\cancel{e^{i\theta}}\, d\theta}{\cancel{2e^{i\theta}}} - 2\int_{\pi/2}^{3\pi/2} \frac{2i\,e^{i\theta}\, d\theta}{(2e^{i\theta})^4}$$

$$= i\cdot\left(\frac{3\pi}{2} - \pi/2\right) - 2\cdot\frac{i}{2^3}\int_{\pi/2}^{3\pi/2} d\theta\ e^{-3i\theta}$$

$$= i\cdot\pi - \frac{i}{2^2}\cdot\left(-\frac{1}{3i}\right)\cdot e^{-3i\theta}\Big|_{\theta=\pi/2}^{3\pi/2}, \quad \text{using} \quad \int_{\theta_1}^{\theta_2} d\theta\ e^{\alpha\theta} = \frac{1}{\alpha}\left(e^{\alpha\theta_2} - e^{\alpha\theta_1}\right).$$

So, $I = i\pi + \frac{1}{3\cdot 2^2}\cdot\left(e^{-3i\cdot\frac{3\pi}{2}} - e^{-3i\pi/2}\right)$

$$= i\pi + \frac{1}{3\cdot 2^2}\left(e^{-i\pi/2} - e^{i\pi/2}\right) = i\pi + \frac{1}{3\cdot 2^2}(-2i) = i\pi - \frac{i}{6} = i\left(\pi - \frac{1}{6}\right)$$

Alternative method: Notice that $\frac{1}{z} = \frac{d}{dz}\mathrm{Ln}\, z$, $\frac{1}{z^4} = -\frac{1}{3}\frac{d}{dz}\frac{1}{z^3}$.

So: $I = \mathrm{Ln}\, z\Big|_{z=2i}^{-2i} + \frac{2}{3}\cdot\frac{1}{z^3}\Big|_{z=2i}^{-2i}$ where $2i = 2e^{i\pi/2}$, $-2i = 2e^{i3\pi/2}$

$$\Rightarrow I = \mathrm{Ln}(-2i) - \mathrm{Ln}(2i) + \frac{2}{3}\left[\frac{1}{(-2i)^3} - \frac{1}{(2i)^3}\right] = i\left(\frac{3\pi}{2} - \pi/2\right) + \frac{2}{3}\,\frac{2}{i}\,\frac{1}{2^3} = i\left(\pi - \frac{1}{6}\right)$$

(V). <u>1.</u> $f(z) = \dfrac{1}{z^2-5z+6} = \dfrac{1}{(z-2)(z-3)} = \dfrac{A}{z-2} + \dfrac{B}{z-3}$.

$z_1 = 2,\ z_2 = 3.$

A : Multiply by $z-2$ and take $z \to 2$:

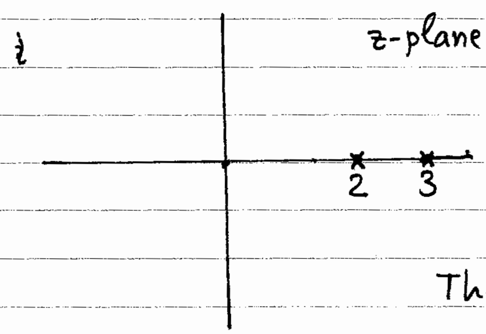
$$A = \lim_{z\to 2}\left[f(z)\cdot(z-2)\right] = \lim_{z\to 2}\frac{1}{z-3} = -1$$

B: Multiply by $z-3$ and let $z \to 3$:

$$B = \lim_{z\to 3}\left[f(z)\cdot(z-3)\right] = \frac{1}{3-2} = 1.$$

So $f(z) = \dfrac{-1}{z-2} + \dfrac{1}{z-3}$

<u>2.</u>



The function ceases to be analytic at $z = 2, 3$.

(i) The region $0 \le |z| < 3$ contains the singular point $z=2$, so we canNOT expand $f(z)$ in Laurent (or Taylor) series in this region

(ii) The region $2 < |z+1| < 3$ does <u>not</u> contain any singular points of $f(z)$, so $f(z)$ is analytic in this region. So, we CAN expand $f(z)$ in Laurent series in this region.

(iii) The region $|z+1| > 3$ contains the singular point $z = 3$, so we canNOT expand $f(z)$ in Laurent (or Taylor) series in this region beware!

3. Let $w = z-2 \Rightarrow z = 2+w$, where $0<|w|<1$

$$f(z) = \frac{1}{w.(w-1)} = \frac{-1}{w}\,\underbrace{\frac{1}{1-w}}_{\text{our } \lambda!\ |\lambda|<1} = -\frac{1}{w}\sum_{n=0}^{\infty} w^n = -\frac{1}{z-2}\sum_{n=0}^{\infty}(z-2)^n$$

$$\Rightarrow f(z) = -\sum_{n=0}^{\infty}(z-2)^{n-1} = -\frac{1}{z-2} - \underbrace{\sum_{n=1}^{\infty}(z-2)^{n-1}}_{\text{non-negative powers of } z-2}$$

This is a Laurent series for $f(z)$, convergent for $0<|z-2|<1$.

exclude 0!

VI. $$f(z) = \frac{1}{(z^2+2)\,(z^2+3)}.$$

The singular ("bad") points of this function, where it ceases to be analytic, occur at $z^2+2=0 \Rightarrow z=\pm\sqrt{2}\,i$, $z^2+3=0 \Rightarrow z=\pm\sqrt{3}\,i$.

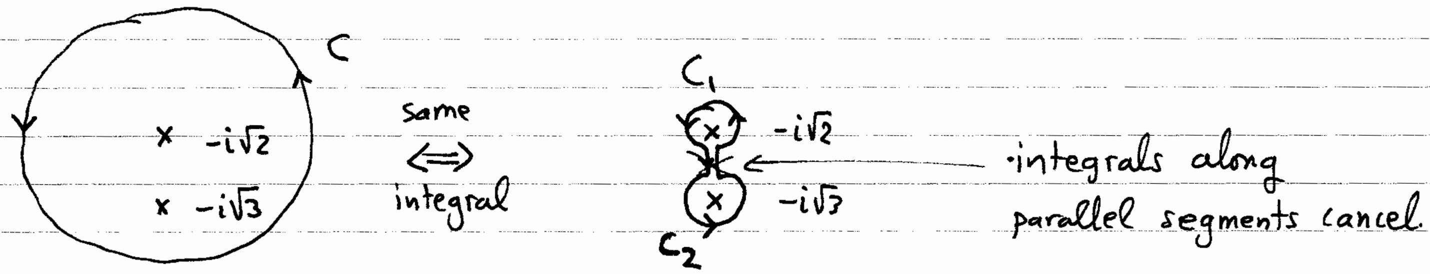
(A) The circle C with center $-i$ and radius $\rho = 1/4$ does NOT contain any singular point of $f(z)$. By the Cauchy integral theorem,

$$\oint_C dz\; f(z) = 0.$$

(B) With $\rho = 1$, the circle contains the singular points $-i\sqrt{2}$, $-i\sqrt{3}$.

Clearly,
$$\oint_C dz\; f(z) = \oint_{C_1} dz\; f(z) + \oint_{C_2} dz\; f(z)$$

where C_1 is a small circle centered at $-i\sqrt{2}$, and C_2 is a small circle centered at $-i\sqrt{3}$. We used the fact that $f(z)$ is analytic in the region between C, C_1 and C_2



We apply the Cauchy integral formula to calculate $\oint_{C_1}$ and $\oint_{C_2}$:

- $I_1 = \oint_{C_1} dz\, f(z) = \oint_{C_1} dz \dfrac{\boxed{\dfrac{1}{(z-i\sqrt{2})(z^2+3)}}}{z+i\sqrt{2}} = \oint_{C_1} dz \dfrac{f_1(z)}{z-z_1}$, $z_1 = -i\sqrt{2}$ inside C_1,

$f_1(z)$: analytic inside C_1.

$\Rightarrow I_1 = 2\pi i \cdot f_1(z_1)$, $f_1(z) = \dfrac{1}{(z-i\sqrt{2})(z^2+3)}$

$\Rightarrow f_1(z_1) = \dfrac{1}{-2i\sqrt{2}\,(3-2)} = \dfrac{i}{2\sqrt{2}} \Rightarrow I_1 = -\dfrac{\pi}{\sqrt{2}}$

- $I_2 = \oint_{C_2} dz \dfrac{f_2(z)}{z-z_2}$, $z_2 = -i\sqrt{3}$, $f_2(z) = \dfrac{1}{(z-i\sqrt{3})(z^2+2)}$

$= 2\pi i \cdot f_2(z_2) = 2\pi i \cdot \dfrac{1}{+2i\sqrt{3}\,(+1)} = \dfrac{\pi}{\sqrt{3}}$.

$$\oint_C dz\, f(z) = I_1 + I_2 = \pi\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}\right)$$

Ⓒ With $\rho = 4$, the circle C contains *all* singular points of $f(z)$ in its interior. So, $f(z)$ is analytic everywhere outside C. Thus, we can modify C taking its radius $\to +\infty$ without changing the result of integration. For large $|z|$ in the integrand

$$\oint_C dz\, f(z) \longrightarrow \oint_{\substack{C \\ \nearrow \text{large circle}}} dz\, \frac{1}{\underbrace{(z^2+\cancel{3})\,(z^2+\cancel{2})}_{z^2 \text{ dominates}}} = \oint_C \frac{dz}{z^4} = 0,$$

because it's an integral $\oint_C dz \cdot z^n$ with $n \neq -1$, where C contains 0.

VII 1. $f(z) = e^{z^2} \sin z$.

$f(z)$ is analytic everywhere. By Cauchy integral theorem,

$$\oint_C dz\, f(z) = 0.$$

2. $f(z) = \dfrac{1}{z^{10}}$. Let $z = e^{i\theta}$, $0 \le \theta < 2\pi$

$$\oint_C \frac{dz}{z^{10}} = \int_0^{2\pi} \frac{i e^{i\theta} d\theta}{e^{i10\theta}} = i \int_0^{2\pi} e^{-i9\theta}\, d\theta = -\frac{i}{i9}\, e^{-i9\theta}\Big|_{\theta=0}^{2\pi} = 0.$$

3. $f(z) = \tan z = \dfrac{\sin z}{\cos z}$; "bad points" at $\cos z = 0 \Rightarrow z = (2n+1)\frac{\pi}{2}$, n: integer

The unit circle ~~with~~ does NOT contain any of these points $\Rightarrow \oint_C dz\, f(z) = 0$ by Cauchy integral theorem.

VIII See Lecture Notes

Basic Steps: Modify contour C to a circle of radius ε around $\alpha = b$. Let $\alpha - b = \varepsilon e^{i\theta}$, $0 \le \theta < 2\pi$.

$$\oint_C \frac{f(\alpha)}{\alpha - b}\, d\alpha = \int_0^{2\pi} \frac{f(b+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, \varepsilon i e^{i\theta}\, d\theta = i \int_0^{2\pi} f(b+\varepsilon e^{i\theta})\, d\theta$$

$$\xrightarrow[\varepsilon \to 0]{} i \cdot 2\pi \cdot f(b)$$

(Explain why it is legitimate to modify C this way.)