18.075 Solutions to Practice Test II for Exam 2 [Please check for errors!]

Solutions here are brief. In the actual exam, develop arguments in some detail.

(I) (1)

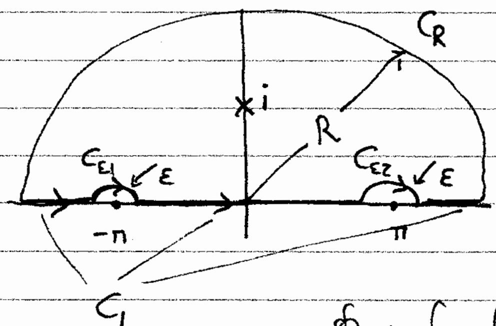
The integrand has singularities at $z=\pm i$ only (at $z=\pm\pi$ the zeros of denominator are canceled by zeros of the numerator). These $z=\pm i$ are simple poles.
So $z=\pm\pi$ are NOT singularities in (1)

(2) The function $\dfrac{e^{iz}}{(z^2-\pi^2)(z^2+1)}$ has simple poles at $z=\pm\pi$ and $z=\pm i$.
[In this case, $z=\pm\pi$ are NOT canceled by numerator].

(3), (4)

$$I = \operatorname{Im} P\int_{-\infty}^{\infty}\frac{x\,e^{ix}}{(x^2-\pi^2)(x^2+1)}\,dx,$$ where principal value $P(\dots)$ is defined as

$$P\int_{-\infty}^{\infty}\frac{x\,e^{ix}}{(x^2-\pi^2)(x^2+1)}\,dx \equiv \lim_{\varepsilon\to0^+}\left(\int_{-\infty}^{-\pi-\varepsilon}+\int_{-\pi+\varepsilon}^{\pi-\varepsilon}+\int_{\pi+\varepsilon}^{\infty}\right)\frac{x\,e^{ix}}{(x^2-\pi^2)(x^2+1)}\,dx \equiv \int_{C_1}\frac{z\,e^{iz}}{(z^2-\pi^2)(z^2+1)}\,dz$$




$$C \equiv C_1 + C_{\varepsilon 1} + C_{\varepsilon 2} + C_R \qquad (\varepsilon\to0^+,\ R\to+\infty)$$

By residue theorem,

- $$\oint_C dz\,\frac{z\,e^{iz}}{(z^2-\pi^2)(z^2+1)} = 2\pi i\,\operatorname*{Res}_{z=i}\left[\frac{z\,e^{iz}}{(z^2-\pi^2)(z^2+1)}\right] = \frac{\cancel{2}\pi i\; i\,e^{-1}}{(-1-\pi^2)}\,\frac{1}{\cancel{2}i} = -i\,\frac{\pi e^{-1}}{\pi^2+1}$$

- $$\oint_C = \int_{C_1}+\int_{C_{\varepsilon 1}}+\int_{C_{\varepsilon 2}}+\int_{C_R}$$

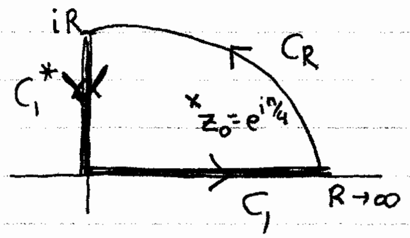
- $$\lim_{\varepsilon\to0^+}\int_{C_{\varepsilon 1}} dz\,\frac{z\,e^{iz}}{(z^2-\pi^2)(z^2+1)} = -\pi i\,\frac{-\pi\,e^{-i\pi}}{-2\pi\,(\pi^2+1)} = +i\,\frac{\pi}{2(\pi^2+1)}$$ (Why?)

- $$\lim_{\varepsilon\to0^+}\int_{C_{\varepsilon 2}} dz\,\frac{z\,e^{iz}}{(z^2-\pi^2)(z^2+1)} = -i\pi\,\frac{\pi\,e^{i\pi}}{2\pi\,(\pi^2+1)} = i\,\frac{\pi}{2(\pi^2+1)}$$ (Why?)

- $$\lim_{R\to\infty}\int_{C_R} dz\,\frac{z\,e^{iz}}{(z^2-\pi^2)(z^2+1)} = 0$$ (Why?)

Hence,
$$I = \operatorname{Im}\left\{-i\,\frac{\pi e^{-1}}{\pi^2+1} - i\,2\,\frac{\pi}{2(\pi^2+1)}\right\} = -\frac{\pi}{\pi^2+1}\,(1+e^{-1})$$

(II) $$I = \int_0^\infty dx \, \frac{x}{x^4+1}$$




Simple poles of integrand:

$$z^4+1=0 \Leftrightarrow z = z_k = e^{i(\pi+2k\pi)/4}, \quad k=0,1,2,3$$

Let $\quad C = C_1 + C_1^* + C_R$

$$\underbrace{\oint_C dz \, \frac{z}{z^4+1}}_{} = 2\pi i \operatorname*{Res}_{z=z_0}\left(\frac{z}{z^4+1}\right) = 2\pi i \, \frac{e^{i\pi/4}}{4e^{i3\pi/4}} = \frac{\pi i}{2} e^{-i\pi/2} = \frac{\pi}{2}$$

$$\oint_C = \int_{C_1} + \int_{C_1^*} + \int_{C_R}$$

- $$\int_{C_1^*} dz \, \frac{z}{z^4+1} \overset{z=iy}{=\!=} -\int_0^\infty i\,dy \, \frac{iy}{y^4+1} = \int_0^\infty dy \, \frac{y}{y^4+1} = I \quad (\text{as } R\to\infty)$$

- $$\lim_{R\to\infty} \int_{C_R} dz \, \frac{z}{z^4+1} = 0 \qquad (\text{Why?})$$

Hence, $$\frac{\pi}{2} = I + I + 0 \iff I = \pi/4.$$

(III) (1) The ratio test, with $A_n(x) = \frac{x^n}{n^n}$, gives

$$\left|\frac{A_{n+1}(x)}{A_n(x)}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)^{n+1}}}{\frac{x^n}{n^n}}\right| = \frac{n^n}{(n+1)^{n+1}}|x| = \frac{1}{n+1}\,\frac{1}{(1+\frac{1}{n})^n}\,|x| \xrightarrow[n\to\infty]{} 0$$

(~~because~~ since $(1+\frac{1}{n})^n \to e$ as $n\to\infty$)

Hence, $\underline{R=\infty}$.

[The root test gives $\sqrt[n]{|A_n(x)|} = \frac{|x|}{n} \to 0$ as $n\to\infty \Rightarrow R=\infty$ ]

(2) With $A_n(x) = \frac{(n)!}{(2n)!}(x\mp 2)$, the ratio test gives

$$\left|\frac{A_{n+1}(x)}{A_n(x)}\right| = \frac{(n+1)!}{n!}\,\frac{(2n)!}{(2n+2)!}\,|x+2| = \frac{n+1}{(2n+1)(2n+2)}\,|x+2| \longrightarrow 0 \quad \text{as } n\to\infty$$

Hence, $\underline{R=\infty}$.

(IV) We write the ode in the form

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

Singular points of the ode are points where $a_1(z)$ and $a_2(z)$ are NOT analytic.

(1) $a_1(x) = x$, $a_2(x) = x^2$ : analytic everywhere

Hence, this ode has no singular points. (all points are ordinary).

(2) $a_1(x) = -\frac{x^2+2}{x^2} = -1 - \frac{2}{x^2}$, $a_2(x) = -\frac{x+1}{x^2} = -\frac{1}{x} - \frac{1}{x^2}$.

$a_1(z) = -1 - \frac{2}{z^2}$ has a double pole at $0 \Rightarrow x_0 = 0$ is a singularity of ode

[ Also: $a_2(z) = -\frac{1}{z} - \frac{1}{z^2}$ has a double pole at $0 \Rightarrow x_0 = 0$ is a singularity of ode. ]

So, for $z_0 = 0$,

$(z-z_0)a_1(z) = -z - \frac{2}{z}$ : has a simple pole at $z_0 = 0$ (NOT analytic)

Hence, $x_0 = 0$ is an irregular singular point of the ode.
(all other points are ordinary)

(V). $Ly \equiv x^2 \frac{d^2y}{dx^2} + (x^2 - x)\frac{dy}{dx} + y = 0$

(1) $y = \sum_{n=0}^{\infty} A_n x^n$, $(x^2-x)\frac{dy}{dx} = (x^2-x)\sum_{n=0}^{\infty} nA_n x^{n-1} = \sum_{n=0}^{\infty} nA_n x^{n+1} - \sum_{n=0}^{\infty} nA_n x^n$

$$= \sum_{n=0}^{\infty}(n-1)A_{n-1}x^n - \sum_{n=0}^{\infty} nA_n x^n; \qquad \underbrace{A_{-1} \equiv 0}$$

$$x^2\frac{d^2y}{dx^2} = x^2\sum_{n=0}^{\infty} n(n-1)A_n x^{n-2} = \sum_{n=0}^{\infty} n(n-1)A_n x^n.$$

Ode gives:

$$\sum_{n=0}^{\infty} n(n-1)A_n x^n + \sum_{n=0}^{\infty}(n-1)A_{n-1}x^n - \sum_{n=0}^{\infty} nA_n x^n + \sum_{n=0}^{\infty} A_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty}\left[ n(n-1)A_n + (n-1)A_{n-1} - nA_n + A_n \right] x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty}\left[ (n^2-2n+1)A_n + (n-1)A_{n-1} \right] x^n = 0.$$

② From last equation, the recurrence relation reads as

$$(n^2-2n+1)\,A_n + (n-1)\,A_{n-1} = 0$$

DON'T CANCEL THIS FACTOR YET!

$$\Leftrightarrow \quad (n-1)^2 A_n + (n-1)\,A_{n-1} = 0 \quad \Leftrightarrow \quad (n-1)\left[(n-1)A_n + A_{n-1}\right] = 0; \qquad A_{-1} \equiv 0,$$

$n = 0, 1, 2, \ldots$

$\underline{n=0}$ : $\quad \underline{A_0 = 0.}$

$\underline{n=1}$ : $\quad 0 = 0 \quad \Rightarrow \quad \underline{A_1 : \text{arbitrary}}.$

$\underline{n \geq 2}$ : $\quad (n-1)\,A_n + A_{n-1} = 0 \quad \Leftrightarrow \quad A_n = -\dfrac{A_{n-1}}{n-1}$

It follows that $A_n$ are given only in terms of $A_1$ while $A_0 = 0$. Hence, the general solution yielded by this method involves only 1 arbitrary constant, $A_1$.

$\Rightarrow$ This method gives only 1 independent solution.