18.075. Solutions to Practice Test 4 for Quiz 3.

Ⓘ ① $$\sum_{n=0}^{\infty} (-1)^n \frac{3^n (x-2)^n}{n!}$$

Let $A_n(x) \equiv (-1)^n \frac{3^n (x-2)^n}{n!}$

Ratio test: $$L = \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = |x-2| \cdot \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)!}}{3^n/n!} \right| = |x-2| \cdot \lim_{n\to\infty} \frac{3}{n+1} = |x-2| \cdot 0$$

So, for every finite x, $L = 0 < 1 \to$ series converges for all x.

② $$\sum_{n=0}^{\infty} a_n x^n, \quad a_n = \begin{cases} n^2+n, & n = 2m \\ 2n^3, & n = 2m+1 \end{cases}$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{2m} x^{2m} + \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1} = \sum_{m=0}^{\infty} \underbrace{[(2m)^2+(2m)]\, x^{2m}}_{\equiv A_m(x)} + \sum_{m=0}^{\infty} \underbrace{2\,(2m+1)^3\, x^{2m+1}}_{\equiv B_m(x)}$$

Ratio test: 1st series: $$L_1 = \lim_{m\to\infty} \left| \frac{A_{m+1}(x)}{A_m(x)} \right| = |x^2| \underbrace{\lim_{m\to\infty} \frac{4(m+1)^2+2(m+1)}{4m^2+2m}}_{=1} = |x^2|$$

So, the 1st series converges for $L_1 < 1 \Leftrightarrow |x|<1$ and diverges for $L_1 > 1 \Leftrightarrow |x|>1$

So, the radius of convergence is $R_1 = 1$.

2nd series: $$L_2 = \lim_{m\to\infty} \left| \frac{B_{m+1}(x)}{B_m(x)} \right| = |x^2| \cdot \underbrace{\lim_{m\to\infty} \frac{2\,(2m+3)^3}{2\,(2m+1)^3}}_{1} = |x^2|$$

So, the 2nd series also has radius of convergence $1 = R_2$.

So, the sum of the two series converges for $R = \min\{R_1, R_2\} = 1$ i.e. for $|x|<1$.

③ $$\sum_{n=0}^{\infty} a_n x^n, \quad a_n = \begin{cases} (3m)^2, & n = 3m \\ 3m+1, & n = 3m+1 \\ 3m+2, & n = 3m+2 \end{cases} \qquad m = 0, 1, 2, \ldots$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{3m} x^{3m} + \sum_{m=0}^{\infty} a_{3m+1} x^{3m+1} + \sum_{m=0}^{\infty} a_{3m+2} x^{3m+2}$$

$$= \sum_{m=0}^{\infty} (9m^2) x^{3m} + \sum_{m=0}^{\infty} (3m+1)\, x^{3m+1} + \sum_{m=0}^{\infty} (3m+2)\, x^{3m+2}$$

By applying the ratio test, we find the radius of convergence of each series to be 1.

Hence, their sum has radius of convergence $= 1$, i.e, $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x|<1$

(II) $x(1-x^2)y'' - (1+x^2)y' + 3xy = 0$

$\Rightarrow (1-x^2)y'' + \frac{1}{x}(-1-x^2)y' + \frac{1}{x^2}3x^2y = 0$; canonical form $\overset{1-x^2}{\overset{\shortparallel}{R(x)}}y'' + \frac{1}{x}\overset{-1-x^2}{\overset{\shortparallel}{P(x)}}y' + \frac{1}{x^2}\overset{3x^2}{\overset{\shortparallel}{Q(x)}}y = 0$

with $R(x) = 1-x^2$ $[R(0) \equiv R_0 = 1]$, $P(x) = -1-x^2$, $Q(x) = 3x^2$

Indicial equation: $0 = f(s) = s(s-1) + \overset{-1}{\overset{\shortparallel}{P_0}}s + \overset{0}{\overset{\shortparallel}{Q_0}} = s(s-1) - s \rightarrow s(s-2) = 0 \rightarrow \left\{ \begin{array}{l} s_2 = 0 \\ s_1 = 2 \end{array} \right.$

Note: $s_1 - s_2 = 2$: integer. $\Rightarrow$ We may or may not have an exceptional case for $s=s_2$
(exceptional case: no solution for $s=s_2$).

Recursive function:

$g_n(s) := R_n(s-n)(s-n-1) + P_n(s-n) + Q_n, \quad n \geq 1$

Clearly, $g_n(s) = 0$ unless $n=2$

$g_2(s) = -(s-2)(s-3) + (-1)(s-2) + 3 = -(s-2)^2 + 3$

Recursive formula: $\overbrace{f(s+k)}^{(s+k)(s+k-2)} A_k = -\sum_{n=1}^{k} g_n(s+k)A_{k-n}$, $k = 1, 2, \ldots$

k = 1 : $(s+1)(s-1)A_1 = 0 \Rightarrow A_1 = 0$, since $\overbrace{(s+1)(s-1)}^{\text{on LHS}} \neq 0$ for $s = s_1$ or s_2

k ≥ 2 : $(s+k)(s+k-2)A_k = -g_2(s+k)A_{k-2}$

$\rightarrow (s+k)(s+k-2)A_k = [(s+k-2)^2 - 3]A_{k-2}$, $s = s_1 = 2$ or $s = s_2 = 0$.

We check how many solutions we get for $s=s_2=0$; we need to examine the recursive formula for $k = s_1 - s_2 = 2$:

k=2 : $0 \cdot A_2 = -3A_0 \neq 0$

So, it is impossible to get any solutions for $s=s_2=0$. by the Frobenius method.

So, we get only 1 independent solution, for $s=s_1=2$

$s=s_1=2$:

k=1 : $A_1 = 0$

k≥2 : $k(k+2)A_k = (k^2-3)A_{k-2} \Rightarrow A_k = \frac{k^2-3}{k(k+2)}A_{k-2}$, $k = 2, 3, \ldots$

It follows that $A_1 = A_3 = \cdots = A_{2m+1} = 0$, $m=0,1,2,\dots$

while $A_{2m} = \dfrac{4m^2-3}{2m(2m+2)} A_{2(m-1)}$

$$\left\{\begin{array}{l} 2\cdot 4\ \cancel{A_2} = (2^2-3)A_0 \\ 4\cdot 6\ A_4 = (4^2-3)\cancel{A_2} \\ \qquad\vdots \\ 2m(2m+2)A_{2m} = [(2m)^2-3]\cancel{A_{2(m-1)}} \end{array}\right. \rightarrow 2\cdot4^2\cdot6^2\cdots(2m)^2(2m+2)A_{2m} = (2^2-3)\dots[(2m)^2-3]A_0$$

$$\Longrightarrow \quad A_{2m} = \frac{(2^2-3)\dots(4m^2-3)}{2^2\cdot[\underset{(2\cdot2)}{\overset{}{4}}\cdot\underset{(2\cdot3)\cdots}{6}\cdots(2m)]^2(m+1)} A_0 \ ; \qquad m \geq 1$$

So, $y_1(x) = x^{s_1}\sum_{n=0}^{\infty} A_n x^n = x^2 \sum_{m=0}^{\infty} A_{2m}x^{2m}$

$$= A_0 x^2\left[1+\sum_{m=1}^{\infty}\frac{(2^2-3)\dots(4m^2-3)}{2^{2m}(m!)^2(m+1)}x^{2m}\right].$$

Ⓘ 1. $xy'' + (1+4x^2)y' + (3x+\lambda x^3)y = 0$

$\longrightarrow \boxed{x^2y''+x(1+4x^2)y'+(3x^2+\lambda x^4)y=0}$ (1)

(2)

We compare this ODE with: $\boxed{x^2y''+x[(1-2A)+2rBx^r]y'+[A^2-p^2s^2+s^2C^2x^{2s}-rB(2A-r)x^r + r^2B^2x^{2r}]y=0}$

Coefficient of y'

$1-2A=1 \Rightarrow \boxed{A=0}$, $\boxed{r=2}$

$2rB=4 \Rightarrow \boxed{B=1}$

Coefficient of y

The coefficient of y in (2) is

$-p^2s^2+s^2C^2x^{2s}+4x^2+4x^4$

$\Rightarrow \boxed{\lambda=4}$ ***

Get $3x^2$: $\boxed{s=1}$, $s^2C^2+4=3 \Rightarrow C^2=-1 \Rightarrow \boxed{C=i}$

$-p^2s^2=0 \rightarrow \boxed{p=0}$

General solution: $y(x) = \overset{x^A e^{-Bx^r}}{\overset{\shortparallel}{g(x)}} \, Z_p[\overset{Cx^s}{\overset{\shortparallel}{f(x)}}] = e^{-x^2} Z_0(ix)$

$$\Rightarrow \quad y(x) = e^{-x^2}[c_1 I_0(x) + c_2 \cdot K_0(x)] \quad , \quad c_1, c_2 : \text{const.}$$

with $\boxed{\lambda = 4}$

② $y(0) = -2$ (: finite) $\Rightarrow$ $\underline{c_2 = 0}$ (because $K_0(x)$ blows up at $x=0$!)

$$\Rightarrow y(x) = e^{-x^2} c_1 I_0(x)$$

$$y(0) = -2 \rightarrow e^{-0} c_1 \underbrace{I_0(0)}_{1} = -2 \rightarrow c_1 = -2$$

$$\rightarrow \quad y(x) = -2 e^{-x^2} I_0(x)$$

(IV) $$x^2 \frac{d^2y}{dx^2} + x(3+x)\frac{dy}{dx} + (-3+\lambda)y = 0.$$

Of form: $a_0(x)\, y'' + a_1(x)\, y' + [a_2(x) + \lambda a_3(x)]\, y = 0.$; $a_0 = x^2$, $a_1 = x(3+x)$, $a_2 = -3$, $a_3 = 1$.

$$p(x) = e^{\int \frac{a_1}{a_0} dx} = e^{\int \frac{x(3+x)}{x^2} dx} = e^{3\ln x + x} = x^3 e^x \, ,$$

$$q(x) = \frac{a_2}{a_0} p = \frac{-3}{x^2} x^3 e^x = -3x e^x \, ,$$

$$r(x) = \frac{a_3}{a_0} p = \frac{1}{x^2} x^3 e^x = x e^x .$$

<u>Check</u>: $\frac{d}{dx}\left[p(x) \frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y =$

$$= (x^3 e^x y')' + (-3xe^x + \lambda x e^x)\, y = x^3 e^x y'' + (3x^2 e^x + x^3 e^x)\, y' + (-3xe^x + \lambda x e^x)\, y$$

$$= x^3 e^x \, y'' + x^2 e^x (3+x)\, y' + x e^x (-3+\lambda)\, y$$

$$= x e^x \left\{ x^2 y'' + x(3+x) y' + (-3+\lambda) y \right\} = 0 \qquad \checkmark$$

Ⓥ Solve $\overbrace{x^2y'' + xy' + \lambda x^2 y = 0}^{\text{Bessel eqn.}}$, $y(0)=1$, $y(A)=0$.

General solution: $y(x) = c_1 \overbrace{J_0(\sqrt{\lambda}\, x) + c_2 \cdot Y_0}^{\text{Bessel fcns}}(\sqrt{\lambda}\, x)$

- $y(0) = 1$ (: finite) $\Rightarrow$ $\underline{c_2 = 0}$ (since $Y_0(x)$ blows up at $x=0$)

Then $y(x) = c_1 J_0(\sqrt{\lambda}\, x)$; $y(0)=1 \Rightarrow \boxed{c_1 = 1}$, since $J_0(0)=1$

So, $y(x) = J_0(\sqrt{\lambda}\, x)$

- $y(A) = 0 \rightarrow J_0(\sqrt{\lambda}\, A) = 0 \rightarrow \lambda = \lambda_n = (\xi_n / A)^2$, ξ_n: zeros of $J_0(x)=0$, $J_0(\xi_n)=0$, $n = 1, 2, \ldots$

Hence, $y(x) = \phi_n(x) = J_0\left(\frac{\xi_n x}{A}\right)$, $n = 1, 2, \ldots$

The given problem can be thought of as a Sturm-Liouville problem,

$$\left\{ \begin{array}{l} [p(x)y'(x)]' + [q(x) + \lambda r(x)]y(x) = 0 \quad , \quad 0 \le x \le A \\ \underbrace{y(0) : \text{finite}, \quad y(A) = 0}_{\text{homog. b.c's}} \end{array} \right.$$

with $p(x) = x$, $q(x) = 0$, $r(x) = x$.

For any φ_1, φ_2 that solve this problem, with $\lambda_1 \neq \lambda_2$,

$$\int_0^A dx \; r(x)\, \varphi_1(x)\, \varphi_2(x) = 0$$: orthogonality with weighting function $r(x)$.

$$\Rightarrow \int_0^A dx \; x\, J_0(\sqrt{\lambda_1}\, x)\, J_0(\sqrt{\lambda_2}\, x) = 0.$$

<u>Note</u> : The given condition $y(0) = 1$ determines the arbitrary constant that multiplies $\phi(x)$ in the solution of the Sturm-Liouville problem. That constant is immaterial. What matters is that $y(0)$: finite and NOT the specific value of this <u>finite</u> part (in order to get the orthogonality condition).

ⓋⅠ If $\lambda_n \neq \lambda_p$ we deduce that $\int_a^b dx \; r(x)\, \phi_n(x)\, \phi_p(x) = 0.$

[Easy!]

XII ① $f(x) = e^x + 1 \quad , \quad 0 < x < \pi$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) \, ,$$

$$A_0 = \frac{1}{\pi} \int_0^{\pi} dx \, (e^x + 1) = \frac{1}{\pi} \left(e^x \Big|_0^{\pi} + \pi \right) = \frac{1}{\pi} \left(e^{\pi} - 1 + \pi \right),$$

$$A_{n \geq 1} = \frac{2}{\pi} \int_0^{\pi} dx \, (e^x + 1) \cdot \cos(nx) = \frac{2}{\pi} \left[\int_0^{\pi} dx \;\; e^x \cos(nx) + \frac{1}{n} \overset{\to 0}{\cancel{\sin(nx) \Big|_0^{\pi}}} \right] ;$$

Evaluate:

$$I = \int_0^{\pi} dx \;\; e^x \cos(nx) = \int_0^{\pi} d(e^x) \cdot \cos(nx) = e^x \cos(nx) \Big|_0^{\pi} + n \int_0^{\pi} dx \;\; e^x \sin(nx)$$

$$= e^{\pi}(-1)^n - 1 + n \overset{\to 0}{\cancel{e^x \sin(nx) \Big|_0^{\pi}}} - n^2 \underbrace{\int_0^{\pi} dx \;\; e^x \cos(nx)}_{I} \Rightarrow I(1 + n^2) = (-1)^n e^{\pi} - 1$$

$$\rightarrow \boxed{I = \frac{(-1)^n e^{\pi} - 1}{1 + n^2}}$$

So, $A_{n \geq 1} = \frac{2}{\pi} \; \frac{(-1)^n e^{\pi} - 1}{1 + n^2}$

$$f(x) = \frac{1}{\pi} \left(e^{\pi} - 1 + \pi \right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi} - 1}{1 + n^2} \; \cos(nx)$$

② $f'(x) = e^x = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi} - 1}{1 + n^2} \; n \sin(nx)$

This series still converges (conditionally). Hence, the RHS gives the Fourier sine series of $f'(x) = e^x$.

③ $$h(x) = \begin{cases} -1, & 0 \le x \le \ell/2 \\ 2, & \ell/2 < x \le \ell. \end{cases}$$

$$h(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right) \quad , \quad B_{n\geq 1} = \frac{2}{\ell}\int_0^{\ell} f(x)\,\sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$\Rightarrow B_{n\geq 1} = \frac{2}{\ell}\left[\int_0^{\ell/2} (-1)\,\sin\left(\frac{n\pi x}{\ell}\right) dx + \int_{\ell/2}^{\ell} 2\,\sin\left(\frac{n\pi x}{\ell}\right) dx\right]$$

$$= \frac{2}{\ell}\left\{\frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^{\ell/2} - 2\frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\Big|_{\ell/2}^{\ell}\right\}$$

$$= \frac{2}{\ell}\left\{\frac{\ell}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] - \frac{2\ell}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]\right\}$$

$$= \frac{2}{\ell}\cdot\frac{\ell}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1 - 2(-1)^n + 2\cos\left(\frac{n\pi}{2}\right)\right] = \frac{2}{n\pi}\left[3\cos\left(\frac{n\pi}{2}\right) - 2(-1)^n - 1\right]$$

VIII ① $f(x) = (\pi - |x|)^2, \quad -\pi < x < \pi$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)\,, \quad -\pi < x < \pi$$

$$A_0 = \frac{1}{\pi}\int_0^{\pi} dx\, f(x) = \frac{1}{\pi}\int_0^{\pi} dx\,(\pi - x)^2 \overset{y=\pi-x}{=\!=} \frac{1}{\pi}\int_0^{\pi} dy\, y^2 = \frac{1}{\pi}\,\frac{\pi^3}{3} = \pi^2/3\,,$$

$$A_{n\geq 1} = \frac{2}{\pi}\int_0^{\pi} dx\, f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\pi} dx\,(\pi - x)^2\cos(nx) \overset{y=\pi-x}{=\!=} \frac{2}{\pi}\int_0^{\pi} dy\, y^2\cdot \underbrace{\cos[\overset{n\pi - ny}{n(\pi - y)}]}_{(-1)^n\cos(ny)}$$

$$= \frac{(-1)^n}{\pi}\,\frac{2}{n}\,\cancel{y^2\sin(ny)\Big|_0^{\pi}} \to 0 - 2\frac{(-1)^n}{n\pi}2\int_0^{\pi} dy\cdot y\,\sin(ny)$$

$$= +4\frac{(-1)^n}{n\pi}\frac{1}{n}\,y\cos(ny)\Big|_0^{\pi} - 4\frac{(-1)^n}{n^2\pi}\cancel{\int_0^{\pi} dy\,\cos(ny)} \to 0 = 4\frac{(-1)^n}{n^2\pi}\,\pi\,(-1)^n = \frac{4}{n^2}.$$

So, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nx). \qquad (1)$$

② Set <u>$x=0$</u> in both sides of Eq. (1). Since $f(x) = (\pi-|x|)^2$ is <u>continuous</u> at $x=0$, the Fourier series converges to $f(0)$.

$$f(0) = \pi^2 = \frac{\pi^2}{3} + \cancel{2}4\sum_{n=1}^{\infty} \frac{\cos(n\cdot 0)}{n^2} = \frac{\pi^2}{3} + \cancel{2}4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = \cancel{2}4\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$