NAME:

18.075 In–class Exam # 3
December 8, 2004

Justify your answers. Cross out what is not meant to be part of your final answer.
Total number of points: 67.5

I. Consider the ODE

$$xy'' - xy' - y = 0 \qquad (1).$$

1.(2 pts) Classify the point $x_0 = 0$.

2.(2 pts) Write the ODE in the canonical form

$$R(x)y'' + \frac{1}{x}\,P(x)y' + \frac{1}{x^2}\,Q(x)y = 0.$$

3.(3 pts) Find the indicial exponents s_1 and s_2 for this ODE.

4.(8 pts) Obtain a solution to this ODE by the Frobenius method for the largest of the two exponents, s_1.

5.(5 pts) How many independent solutions can you find by repeating part (4) for $s = s_2$?

6.(4 pts) Give the form of the general solution to the ODE (1) of page 1.

II. Consider the Bessel equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad (2).$$

1.(4 pts) Find the general solution to this ODE in terms of elementary functions when $p = 1/2$.

2.(5 pts) Determine the solution $y(x)$ that satisfies $y(\pi) = 0$ and $y'(\pi) = 1$ when $p = 1/2$.

3.(4 pts) Find the general solution to the ODE (2) of page 5 when $p = 2$.

4.(5 pts) Determine the solution $y(x)$ that satisfies $\lim_{x \to 0} y(x) = 0$ when $p = 2$. Is this solution unique? Explain.

III. Solve the following ODEs in terms of Bessel functions, or elementary functions, if possible. DO NOT use the Frobenius method.

1.(4 pts) $xy'' - 9y' + xy = 0$

2.(4 pts) $xy'' + (1 + 6x^2)y' + x(2 + 9x^2)y = 0$

IV. 1. (4.5 pts) Find the Fourier sine series expansion in $[0, \pi]$ of the function defined in $[0, \pi]$ by: $h(x) = -1$ in $[0, \pi/2]$ and $h(x) = 1$ in $(\pi/2, \pi]$. Sketch the function represented by the sine series in the symmetric interval $[-\pi, \pi]$.

2. (6 pts) Find the Fourier cosine series expansion in $[0, \pi]$ of the function $h(x)$ defined in part (1) of previous page. Sketch the function represented by this cosine series in the symmetric interval $[-\pi, \pi]$.

3. (7 pts) Consider the ODE

$$\frac{d^2y}{dx^2} + p^2\, y = h(x), \quad 0 < x < \pi,$$

with the boundary conditions $\underline{y(0) = 0 = y(\pi)}$; $y = y(x)$ is unknown and p is real, $p > 0$. Find the solution $y(x)$ of this boundary-value problem as an expansion in suitable Fourier series. For what values of p does the problem have a solution? **Hint:** Look closely at the boundary conditions and decide whether to use a sine or cosine series (with coefficients to be found), and substitute in the ODE.