18.075 Practice Test 3 for Quiz 3

December 3, 2004

Justify your answers. Cross out what is not meant to be part of your solution.

Total number of points: 120. Time: 120 min.

I. (20 pts) Starting with the Frobenius series for the Bessel functions, show that:

(a) $J_0'(x) = -J_1(x)$, (b) $\frac{d}{dx}[x J_1(x)] = x J_0(x)$.

II. (20 pts) Use the relations of Prob. I in order to evaluate the following integrals.

(a)

$$\int_0^1 dx\, J_0(x)\, J_1(x).$$

Is the result positive or negative?

(b)

$$\int_0^1 dx\, x^3 J_0(x).$$

III. (20 pts) A scientist tries to make up a model that describes the *exponential* growth of a bacterium population, described by the quantity $y(t)$ as a function of time t ($t \geq 0$). He finally comes up with the following ODE for $y(t)$:

$$y''(t) + \frac{1}{t} y'(t) - y(t) = 0,$$

with the initial condition $y(0) = 1$. Find $y(t)$ and explain why this model can indeed describe exponential growth in time.

IV. A rectangular membrane of dimensions $2a$ and $2b$ ($a, b > 0$) vibrates on the (x, y) plane with frequency ω. The deflection $z(x, y)$ of the membrane from the plane is described by the PDE

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + k^2\, z = 0, \quad -a \leq x \leq a,\ -b \leq y \leq b,$$

where $k^2 = \omega^2/c^2$ and c is a constant. This $z(x, y)$ satisfies the boundary conditions

$$z(\pm a, y) = 0, \qquad z(x, \pm b) = 0.$$

(a) (10 pts) Substituting a solution $z(x, y) = X(x)\, Y(y)$ into the PDE, find the ODEs for $X(x)$ and $Y(y)$.

(b) (20 pts) Applying the boundary conditions for z, solve the ODEs for X and Y. What are the characteristic frequencies ω of the membrane?

V. The following problem involves the method of Frobenius to obtain the general solution near $x = 0$ of the ODE

$$x^2y'' + x(x^2 - \lambda)y' + (x^2 + \lambda)y = 0.$$

(a) (2 pts) Write the ODE in its canonical form

$$R(x)y'' + \frac{P(x)}{x}y' + \frac{Q(x)}{x^2}y \ = \ 0.$$

(b) (3 pts) Find the indicial equation, $f(s) = 0$, of the ODE and solve it.

(c) (5 pts) If λ is not an integer, how many solutions can be found by the method of Frobenius and why? What if $\lambda = 1$?

(d) (5 pts) Assume that $\lambda > 1$ is an integer. For the smallest value of s of part (b), write down the formulas for the nonzero functions g_n and the recursion equation for A_n;

(e) (5 pts) How many linearly independent solutions can you find with the method of Frobenius if $\lambda > 1$ is an even integer?

VI. (10 pts) For what characteristic values of the parameter λ is the following ODE a modified Bessel ODE?

$$x^2y'' + x(x^2 - \lambda)y' + (x^2 + \lambda)y = 0.$$