18.075 Practice Test IV for Quiz 3

December 5, 2004

Justify your answers. Cross out what is not meant to be part of your final answer.

Total number of points: 80. Time: 80 min.

I. Find the domain of convergence of the following series:

1. (1 pts)

$$\sum_{n=0}^{\infty}(-1)^n \frac{3^n(x-2)^n}{n!};$$

2. (2 pts)

$$\sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = n^2 + n$ for n even and $a_n = 2n^3$ for n odd.

3. (2 pts)

$$\sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = n^2$ for n divisible by 3 and n otherwise.

II. (10 pts) Use the method of Frobenius to obtain the general solution of the following ODE, near $x = 0$:

$$x(1-x^2)y'' - (1+x^2)y' + 3xy = 0.$$

How many linearly independent solutions can you find? Why?

III. 1. (10 pts) Let λ be a real parameter. Find the values of λ such that the following ODE can be solved by transforming it to a Bessel equation:

$$xy'' + (1+4x^2)y' + (3x + \lambda x^3)y \;=\; 0.$$

Hint: The solution $y(x)$ should involve modified Bessel functions.

2. (5 pts) For the values of λ of part (1), solve the ODE with the condition $y(0) = -2$.

IV. (5 pts) Write the ODE

$$x^2\frac{d^2y}{dx^2} + x(3+x)\frac{dy}{dx} - 3y + \lambda y \;=\; 0$$

in the standard form

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy + \lambda r y = 0.$$

V. (5 pts) By considering the characteristic functions of the problem

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \lambda xy \ = \ 0, \qquad 0 \le x \le A,$$

with the boundary conditions

$$y(0) = 1; \quad y(A) = 0,$$

show that, if λ_1 and λ_2 are two different characteristic values, then

$$\int_0^A xJ_0(\sqrt{\lambda_1}x)J_0(\sqrt{\lambda_2}x)dx \ = \ 0.$$

VI. (5 pts) A Sturm-Liouville problem on the interval $[a, b]$ has boundary conditions

$$y'(a) = 0 \quad \text{and} \quad y(b) = 0.$$

What can one deduce about any two characteristic functions ϕ_n and ϕ_p?

VII. 1. (10 pts) Find the Fourier cosine series of $f(x) = e^x + 1$ in $(0, \pi)$.
2. (5 pts) Can you differentiate the series of part (1) and obtain the Fourier sine series expansion of $f'(x) = e^x$?
3. (5 pts) Find the Fourier sine series expansion in $(0, l)$ of the function: $h(x) = -1$ in $[0, l/2]$ and $h(x) = 2$ in $(l/2, l]$.

VIII. 1. (10 pts) If $f(x) = (\pi - |x|)^2$ for $-\pi < x < \pi$, expand $f(x)$ in cosine series.
2. (5 pts) From the series of part (1), deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$