Ordinary Differential Equations

Unknown function $y = y(x)$ ← independent variable

ODE: $F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x) = 0$ $\qquad y^{(n)} = \frac{d^n y}{dx^n}$

n: order of ODE

ODE is linear if F is linear in $y^{(m)}$, $m = 0, 1, 2, \ldots, m$

ex linear, 2nd order ODE $\qquad C_2(x)y'' + C_1(x)y' + C_0(x)y = 0$ ← homogeneous; $y=0$ is a solution

$g(x) \neq 0$ ← non-homogeneous

ex $y'' - y = 0$ $\qquad$ 2nd order, linear w/ constant coefficients

homogeneous

Method I $y = e^{rx}$

$y' = re^{rx}, \; y'' = r^2 e^{rx}$

$r^2 - 1 = 0 \rightarrow r = -1, 1 \qquad e^{-x} = 1, \; e^x$

this method works only when the coefficients are constant

general solution

$y(x) = \underbrace{K_1 e^{-x} + K_2 e^x}_{\text{arbitrary constants}}$

$$\boxed{\begin{array}{l} \text{An ODE } A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_1(x)y' + A_0(x)y = 0 \text{ has} \\ n \text{ independent solutions: } y = u_1(x), u_2(x), \ldots, u_n(x) \\ \\ \text{General solution: } y(x) = K_1 u_1(x) + K_2 u_2(x) + \ldots + K_n u_n(x) \end{array}}$$

$u_1(x) = e^{-x}, \quad u_2(x) = e^x \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$

$y = k_1 e^{-x} + k_2 e^x$

$= k_1(\cosh x - \sinh x) + k_2(\cosh x + \sinh x)$

$= K_1' \cosh x + k_2' \sinh x$

Try to find solutions $y(x)=\sum_{n=0}^{\infty} a_n x^n$ power series

$y(x)=a_0+a_1x+a_2x^2+\dots$

find $\underbrace{\text{coefficients } a_n}_{\text{not unique}}$ by substitution in ODE.

ex 1 (cont)

Method II: power series

$$y(x)=\sum_{n=0}^{\infty} a_n x^n \quad a_n\text{: to be found}$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2=m} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$n=m+2$

$$\text{ODE: } \sum_{n=0}^{\infty}\left\{(n+2)(n+1)a_{n+2}-a_n\right\}x^n=0 \quad \text{for all } x \text{ in } (-\delta,\delta)$$

Digression: $\sum_{n=0}^{\infty} K_n x^n = 0$ for all x

$=K_0+K_1x+\dots K_nx^n=0$

$x=0$: $K_0=0$ $\rightarrow K_1x+\dots K_nx^n+\dots=0$

$x(K_1+K_2x+\dots K_nx^{n-1})=0$ $K_1=0$ .. etc $\Rightarrow$ all $K_n=0$

$(n+2)(n+1)a_{n+2}=a_n$ $n=0,1,2,\dots$

$n=0$: $2\cdot1\cdot a_2=a_0$

$n=1$: $3\cdot2\cdot a_3=a_1$

$n=2$: $4\cdot3\cdot a_4=a_2$

$n=3$: $5\cdot4\cdot a_5=a_3$

$n=2k$: $(2k+2)(2k+1)a_{2k+2}=a_{2k}$

$n=2k+1$: $(2k+3)(2k+2)a_{2k+3}=a_{2k+1}$

$2(4\cdot3)\cdots[(2k+2)(2k+1)]a_{2k+2}=a_0$

$\rightarrow a_{2k+2}=\dfrac{a_0}{(2k+2)!}$ $a_{2k+3}=\dfrac{a_1}{(2k+3)!}$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{\substack{k=0 \\ n=2k}}^{\infty} a_{2k} = \sum_{k=0}^{\infty} \frac{a_0 x^{2k}}{(2k)!} + \sum_{\substack{k=0 \\ n=2k+1}}^{\infty} \frac{a_1 x^{2k+1}}{(2k+1)!}$$

$$= a_0\underbrace{\left(1 + \tfrac{1}{2}x^2 + \dots + \tfrac{1}{(2k!)}x^{2k} + \dots\right)}_{\cosh x} + a_1\underbrace{\left(x + \tfrac{1}{3!}x^3 + \dots \tfrac{1}{(2k+1)}x^{2k+1}\right)}_{\sinh x}$$

$$= a \underbrace{\cosh x}_{\frac{e^x+e^{-x}}{2}} + a_1 \underbrace{\sinh x}_{\frac{e^x-e^{-x}}{2}} = \underbrace{\left(\frac{a_0+a_1}{2}\right)}_{c_1} e^x + \underbrace{\left(\frac{a_0+a_1}{2}\right)}_{c_2} e^{-x}$$

ex $x^2 y'' + (x^2+x)y' - y = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\prime}$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$x^2 y' = \sum_{n=0}^{\infty} n a_n x^{\overset{m}{n+1}} = \sum_{m=1,0}^{\infty} (m-1) a_{m-1} x^m \; ; \; \boxed{a_{-1} = 0}$$

$$x y' = \sum_{n=0}^{\infty} n a_n x^n$$

$$x^2 y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-\cancel{2}}$$

ODE: $\sum_{n=0}^{\infty} \{ n(n-\cancel{1})a_n + (n-1)a_{n-1} + \cancel{n a_n} - a_n \} x^n = 0$

$$\to \begin{cases} (n-1)[(n+1)a_n + a_{n-1}] = 0, & n = 0, 1, \dots \\ a_{-1} = 0 \end{cases}$$

$n=0$: $a_0 = 0$

$n=1$: a_1 : arbitrary

$\underline{n>1}$: $a_n = \frac{-a_{n-1}}{n+1}$ to shrink: multiply!

$$a_n = (-1)^{n-1} \frac{a_1}{3 \cdot 4 \cdots (n+1)!}$$

$$a_n = (-1)^{n-1} \frac{2a_1}{(n+1)!}$$

$$y(x) = a_0 + a_1 x + \dots + a_n x^n + \dots = a_1 x + \sum_{n=2}^{\infty} (-1)^n \frac{2a_1}{(n+1)!} x^n$$

$$= a_1 \underbrace{\left[x + \sum_{n=2}^{\infty} (-1)^n \frac{2}{(n+1)!} \right]}_{u(x)}$$

$\underline{\text{one}}$ independent solution. We miss one solution of the ODE.

can show $u(x) = \frac{2}{x}(-1 + x + e^{-x})$