PROFESSOR STRANG: Shall we start? The main job of today is eigenvalues and eigenvectors. The next section in the book and a very big topic and things to say about it. I do want to begin with a recap of what I didn't quite finish last time. So what we did was solve this very straightforward equation. Straightforward except that it has a point source, a delta function. And we solved it, both the fixed-fixed case when a straight line went up and back down and in the free-fixed case when it was a horizontal line and then down with slope minus one. And there are different ways to get to this answer. But once you have it, you can look at it and say, well is it right? Certainly the boundary conditions are correct. Zero slope, went through zero, that's good. And then the only thing you really have to check is does the slope drop by one at the point of the impulse? Because that's what this is forcing us to do. It's saying the slope should drop by one. And here the slope is 1-a going up. And if I take the derivative, it's -a going down. 1-a dropped to -a, good. Here the slope was zero. Here the slope was minus one, good. So those are the right answers. And this is simple, but really a great example.

And then, what I wanted to do was catch the same thing for the matrices. So those matrices, we all know what K is and what T is. So I'm solving, I'm really solving K K inverse equal identity. That's the equation I'm solving. So I'm looking for K inverse and trying to get the columns of the identity. And you realize the columns of the identity are just like delta vectors. They've got a one in one spot, they're a point load just like this thing. So can I just say how I remember K inverse? I finally, you know-- again there are different ways to get to it. One way is MATLAB, just do it. But I guess maybe the whole point is, the whole point of these and the eigenvalues that are coming too, is this. That we have here the chance to see important special cases that work out. Normally we don't find the inverse, print out the inverse of a matrix. It's not nice. Normally we just let eig find the eigenvalues. Because that's an even worse calculation, to find eigenvalues, in general. I'm talking here about our matrices of all sizes n by n. Nobody finds the eigenvalues by hand of n by n matrices. But these have terrific eigenvalues and especially eigenvectors.

So in a way this is a little bit like, typical of math. That you ask about general stuff or you write the equation with a matrix A. So that's the general information. And then there's the specific, special guys with special functions. And here there'll be sines and cosines and exponentials. Other places in applied math, there are Bessel functions and Legendre functions. Special guys. So here, these are special. And how do I complete K inverse? So this four, three, two, one.

Let me complete T inverse. You probably know T inverse already. So T, this is, four, three, two, one, is when the load is way over at the far left end and it's just descending. And now I'm going to-- Let me show you how I write it in. Pay attention here to the diagonal. So this will be three, three, two, one. Do you see that's the solution that's sort of like this one? That's the second column of the inverse so it's solving, I'm solving, T T inverse equals I here. It's the-- The second column is the guy with a one in the second place. So that's where the load is, in position number two. So I'm level, three, three, up to that load. And then I'm dropping after the load. What's the third column of T inverse? I started with that first column and I knew that the answer would be symmetric because T is symmetric, so that allowed me to write the first row. And now we can fill in the rest. So what do you think, if the point load is-- Now, I'm looking at the third column, third column of the identity, the load has moved down to position number three. So what do I have there and there? Two and two. And what do I have last? One. It's dropping to zero. You could put zero in green here if you wanted. Zero is the unseen last boundary, you know, row at this end. And finally, what's happening here? What do I get from that? All one, one, one to the diagonal. And then sure enough it drops to zero. So this would be a case where the load is there. It would be one, one, one, one and then boom. No, it wouldn't be. It'd be more like this. One, one, one, one and then down to--

Okay. That's a pretty clean inverse. That's a very beautiful matrix. Don't you admire that matrix? I mean, if they were all like that, gee, this would be a great world. But of course it's not sparse. That's why we don't often use the inverse. Because we had a sparse matrix T that was really fast to compute with. And here, if you tell me the inverse, you've actually slowed me down. Because you've given me now a dense matrix, no zeroes even and multiplying T inverse times the right side would be slower than just doing elimination.

Now this is the kind of more interesting one. Because this is the one that has to go up to the diagonal and then down. So let me-- can I fill in what I think-- way this one goes? I'm going upwards to the diagonal and then I'm coming down to zero. Remember that I'm coming down to zero on this K. So Zero, zero, zero, zero is kind of the row number. If that's row number zero, here's one, two, three, four, the real thing. And then row five is getting back to zero again. So what do you think, finish the rest of that column. So you're telling me now the response to the load in position two. So it's going to look like this. In fact, it's going to look very like this. There's the three and then this is in position two. And then I'm going to have something here and something here and it'll drop to zero. What do I get? Four, two. Six, four, two, zero. It's dropping to zero. I'm going to finish this in but then I'm going to look back and see have I really got it right. How does this go now? Two, let's see. Now it's going up from zero to two to four to six. That's on the diagonal. Now it starts down. It's got to get to zero, so that'll be a three. Here is a one going up to two to three to four. Is that right? And then dropped fast to zero. Is that correct? Think so, yep. Except, wait a minute now. We've got the right overall picture. Climbing up, dropping down. Climbing up, dropping down. Climbing up, dropping down. All good.

But we haven't yet got, we haven't checked yet that the change in the slope is supposed to be one. And it's not. Here the slope is like, three, It's going up by threes and then it's going down by twos. So we've gone from going up at a slope of three to down to a slope of two. Up three, down just like this. But that would be a change in slope of five. Therefore there's a 1/5. So this is going up with a slope of four and down with a slope of one. Four dropping to one when I divide by the five, that's what I like. Here is up by twos, down by threes, again it's a change of five so I need the five. Up by ones, down by four. Sudden, that's a fast drop of four. Again, the slope changed by five, dividing by five, that's got it. So that's my picture. You could now create K inverse for any size. And more than that, sort of see into K inverse what those numbers are. Because if I wrote the five by five or six by six, doing it a column at a time, it would look like a bunch of numbers. But you see it now. Do you see the pattern? Right. This is one way to get to those inverses, and homework problems are offering other ways. T, in particular, is quite easy to invert.

Do I have any other comment on inverses before the lecture on eigenvalues really starts? Maybe I do have one comment, one important comment. It's this, and I won't develop it in full, but let's just say it. What if the load is not a delta function? What if I have other loads? Like the uniform load of all ones or any other load? What if the discrete load here is not a delta vector? I now know the responses to each column of the identity, right? If I put a load in position one, there's the response. If I put a load in position two, there is the response. Now, what if I have other loads? Let me take a typical load. What if the load was, well, the one we looked at before. If the load was [1, 1, 1, 1]. So that I had, the bar was hanging by its own weight, let's say. In other words, could I solve all problems by knowing these answers? That's what I'm trying to get to. If I know these special delta loads, then can I get the solution for every load? Yes, no? What do you think? Yes, right. Now with this matrix it's kind of easy to see because if you know the inverse matrix, well you're obviously in business. If I had another load, say another load f for load, I would just multiply by K inverse, no problem. But I want to look a little deeper. Because if I had other loads here than a delta function, obviously if I had two delta functions I could just combine the two solutions. That's linearity that we're using all the time. If I had ten delta functions I could combine them. But then suppose I had instead of a bunch of spikes, instead of a bunch of point loads, I had a distributed load. Like all ones, how could I do it?

Main point is I could. Right? If I know these answers, I know all answers. If I know the response to a load at each point, then-- come back to the discrete one. What would be the answer if the load was [1, 1, 1, 1]? Suppose I now try to solve the equation Ku=ones(4,1), so all ones. What would be the answer? How would I get it? I would just add the columns. Now why would I do that? Right. Because this, the right-hand side, the input is the sum of the four columns, the four special inputs. So the output is the sum of the four outputs, right. In other words, as you saw, we must know everything. And that's the way we really know it. By linearity. If the input is a combination of these, the output is the same combination of those. Right. So, for example, in this T case, if input was, if I did Tu=ones, I would just add those and the output would be [10 9, 7, 4]. That would be the output from [1, 1, 1, 1].

And now, oh boy. Actually, let me just introduce a guy's name for these solutions and not today show you. You have the idea, of course. Here we added because everything was discrete. So you know what we're going to do over here. We'll take integrals, right? A general load will be an integral over point loads. That's the idea. A fundamental idea. That some other load, f(x), is an integral of these guys. So the solution will be the same integral of these guys. Let me not go there except to tell you the name, because it's a very famous name. This solution u with the delta function is called the Green's function. So I've now introduced the idea, this is the Green's function. This guy is the Green's function for the fixed-fixed problem. And this guy is the Green's function for the free-fixed problem. And the whole point is, maybe this is the one point I want you to sort of see always by analogy. The Green's function is just like the inverse.

What is the Green's function? The Green's function is the response at x, the u at x, when the input, when the impulse is at a. So it sort of depends on two things. It depends on the position a of the input and it tells you the response at position x. And often we would use the letter G for Green. So it depends on x and a. And maybe I'm happy if you just sort of see in some way what we did there is just like what we did here. And therefore the Green's function must be just a differential, continuous version of an inverse matrix.

Let's move on to eigenvalues with that point sort of made, but not driven home by many, many examples. Question, I'll take a question, shoot. Why did I increase zero, three, six and then decrease six? Well intuitively it's because this is copying this. What's wonderful is that it's a perfect copy. I mean, intuitively the solution to our difference equation should be like the solution to our differential equation. That's why if we have some computational, some differential equation that we can't solve, which would be much more typical than this one, that we couldn't solve it exactly by pencil and paper, we would replace derivatives by differences and go over here and we would hope that they were like pretty close. Here they're right, they're the same. Oh the other columns? Absolutely. These guys? Zero, two, four, six going up. Six, three, zero coming back. So that's a discrete thing of one like that. And then the next guy and the last guy would be going up one, two, three, four and then sudden drop.

Thanks for all questions. I mean, this sort of, by adding these guys in, the first one actually went up that way. You see the Green's functions. But of course this has a Green's function for every a. x and a are running all the way from zero to one. Here they're just discrete positions. Thanks. So playing with these delta functions and coming up with this solution, well, as I say, different ways to do it. I worked through one way in class last time. It takes practice. So that's what the homework's really for. You can see me come up with this thing, then you can, with leisure, you can follow the steps, but you've gotta do it yourself to see.

Eigenvalues and, of course, eigenvectors. We have to give them a fair shot. Square matrix. So I'm talking about general, what eigenvectors and eigenvalues are and why do we want them. I'm always trying to say what's the purpose, you know, not doing this just for abstract linear algebra. We do this, we look for these things because they tremendously simplify a problem if we can find it. So what's an eigenvector? The eigenvalue is this number, lambda, and the eigenvector is this vector y. And now, how do I think about those? Suppose I take a vector and I multiply by A. So the vector is headed off in some direction. Here's a vector v. If I multiply, and I'm given this matrix, so I'm given the matrix, whatever my matrix is. Could be one of those matrices, any other matrix. If I multiply that by v, I get some result, Av. What do I do? I look at that and I say that v was not an eigenvector. Eigenvectors are the special vectors which come out in the same direction. Av comes out parallel to v. So this was not an eigenvector.

Very few vectors are eigenvectors, they're very special. Most vectors, that'll be a typical picture. But there's a few of them where I've a vector y and I multiply by A. And then what's the point? Ay is in the same direction. It's on that same line as y. It could be, it might be twice as far out. That would be Ay=2y. It might go backwards. This would be a possibility, Ay=-y. It could be just halfway. It could be, not move at all. That's even a possibility. Ay=0y. Count that. Those y's are eigenvectors and the eigenvalue is just, from this point of view, the eigenvalue has come in second because it's-- So y was a special vector that kept its direction. And then lambda is just the number, the two, the zero, the minus one, the 1/2 that tells you stretching, shrinking, reversing, whatever. That's the number. But y is the vector. And notice that if I knew y and I knew it was an eigenvector, then of course if I multiply by A, I'll learn the eigenvalue. And if I knew an eigenvalue, you'll see how I could find the eigenvector. Problem is you have to find them both. And they multiply each other. So we're not talking about linear equations anymore. Because one unknown is multiplying another. But we'll find a way to look to discover eigenvectors and eigenvalues.

I said I would try to make clear what's the purpose. The purpose is that in this direction on this y line, line of multiples of y, A is just acting like a number. A is some big n by n, 1,000 by 1,000 matrix. So a million numbers. But on this line, if we find it, if we find an eigenline, you could say, an eigendirection in that direction, all the complications of A are gone. It's just acting like a number. So in particular we could solve 1,000 differential equations with 1,000 unknown u's with this 1,000 by 1,000 matrix. We can find a solution and this is where the eigenvector and eigenvalue are going to pay off. You recognize this. Matrix A is of size 1,000. And u is a vector of 1,000 unknowns. So that's a system of 1,000 equations. But if we have found an eigenvector and its eigenvalue then the equation will, if it starts in that direction it'll stay in that direction and the matrix will just be acting like a number. And we know how to solve u'=lambda*u. That one by one scalar problem we know how to solve. The solution to that is e to the lambda*t. And of course it could have a constant in it. Don't forget that these equations are linear. If I multiply it, if I take 2e^(lambda*t), I have a two here and a two here and it's just as good. So I better allow that as well. A constant times e^(lambda*t) times y.

Notice this is a vector. It's a number times a number, the growth. So the lambda is now, for the differential equation, the lambda, this number lambda is crucial. It's telling us whether the solution grows, whether it decays, whether it oscillates. And we're just looking at this one normal mode, you could say normal mode, for eigenvector y. We certainly have not found all possible solutions. If we have an eigenvector, we found that one. And there's other uses and then, let me think. Other uses, what? So let me write again the fundamental equation, Ay=lambda*y. So that was a differential equation. Going forward in time.

Now if we go forward in steps we might multiply by A at every step. Tell me an eigenvector of A squared. I'm looking for a vector that doesn't change direction when I multiply twice by A. You're going to tell me it's y. y will work. If I multiply once by A I get lambda times y. When I multiply again by A I get lambda squared times y. You see eigenvalues are great for powers of a matrix, for differential equations. The nth power will just take the eigenvalue to the nth. The nth power of A will just have lambda to the nth there.

You know, the pivots of a matrix are all messed up when I square it. I can't see what's happening with the pivots. The eigenvalues are a different way to look at a matrix. The pivots are critical numbers for steady-state problems. The eigenvalues are the critical numbers for moving problems, dynamic problems, things are oscillating or growing or decaying. And by the way, let's just recognize since this is the only thing that's changing in time, what would be the-- I'll just go down here, e^(lambda*t). Let's just look and see. When would I have decay? Which you might want to call stability. A stable problem. What would be the condition on lambda to get-- for this to decay. Lambda less than zero. Now there's one little bit of bad news. Lambda could be complex. Lambda could be 3+4i. It can be a complex number, these eigenvalues, even if A is real. You'll say, how did that happen, let me see? I didn't think. Well, let me finish this thought. Suppose lambda was 3+4i. So I'm thinking about what would e to the lambda*t do in that case? So this is small example. If I had lambda is (3+4i), t. What does that do as time grows? It's going to grow and oscillate.

And what decides the growth? The real part. So it's really the decay or growth is decided by the real part. The three, e to the 3t, that would be a growth. Let me put growth. And that would be, of course, unstable. And that's a problem when I have a real part of lambda bigger than zero. And then if lambda has a zero real part, so it's pure oscillation, let me just take a case like that. So e^(4it). So that would be, oscillating, right? It's cos(4t) + i*sin(4t), it's just oscillating.

So in this discussion we've seen growth and decay. Tell me the parallels because I'm always shooting for the parallels. What about the growth of A? What matrices, how can I recognize a matrix whose powers grow? How can I recognize a matrix whose powers go to zero? I'm asking you for powers here, over there for exponentials somehow. So here would be A to higher and higher powers goes to zero, the zero matrix. In other words, when I multiply, multiply, multiply by that matrix I get smaller and smaller and smaller matrices, zero in the limit. What do you think's the test on the lambda now?

So what are the eigenvalues of A to the k? Let's see. If A had eigenvalues lambda, A squared will have eigenvalues lambda squared, A cubed will have eigenvalues lambda cubed, A to the thousandth will have eigenvalues lambda to the thousandth. And what's the test for that to be getting small? Lambda less than one. So the test for stability will be-- In the discrete case, it won't be the real part of lambda, it'll be the size of lambda less than one. And growth would be the size of lambda greater than one. And again, there'd be this borderline case when the eigenvalue has magnitude exactly one. So you're seeing here and also here the idea that we may have to deal with complex numbers here. We don't have to deal with the whole world of complex functions and everything but it's possible for complex numbers to come in.

Well while I'm saying that, why don't I give an example where it would come in. This is going to be a real matrix with complex eigenvalues. Complex lambdas. It'll be an example. So I guess I'm looking for a matrix where y and Ay never come out in the same direction. For real y's I know, okay, here's a good matrix. Take the matrix that rotates every vector by 90 degrees. Or by theta. But let's say here's a matrix that rotates every vector by 90 degrees. I'm going to raise this board and hide it behind there in a minute. I just wanted to-- just to open up this thought that we will have to face complex numbers. If you know how to multiply two complex numbers and add them, you're okay. This isn't going to turn into a big deal. But let's just realize that-- Suppose that matrix, if I put in a vector y and I multiply by that matrix, it'll turn it through 90 degrees. So y couldn't be an eigenvector. That's the point I'm trying to make. No real vector could be the eigenvector of a rotation matrix because every vector gets turned. So that's an example where you'd have to go to complex vectors. and I think if I tried the vector [1, i], so I'm letting the square root of minus one into here, then I think it would come out. If I do that multiplication I get minus i. And I get one. And I think that this is, what is it? This is probably minus i times that. So this is minus i times the input. No big deal. That was like, you can forget that. It's just complex numbers can come in.

Now let me come back to the main point about eigenvectors. Things can be complex. So the main point is how do we use them? And how many are there? Here's the key. A typical, good matrix, which includes every symmetric matrix, so it includes all of our examples and more, if it's of size 1,000, it will have 1,000 different eigenvectors. And let me just say for our symmetric matrices those eigenvectors will all be real. They're great, the eigenvectors of symmetric matrices.

Oh, let me find them for one particular symmetric matrix. Say this guy. So that's a matrix, two by two. How many eigenvectors am I now looking for? Two. You could say, how do I find them? Maybe with a two by two, I can even just wing it. We can come up with a vector that is an eigenvector. Actually that's what we're going to do here is we're going to guess the eigenvectors and then we're going to show that they really are eigenvectors and then we'll know the eigenvalues and it's fantastic. So like let's start here with the two by two case. Anybody spot an eigenvector? Is [1, 0] an eigenvector? Try [1, 0]. What comes out of [1, 0]? Well that picks the first column, right? That's how I see, multiplying by [1, 0], that says take one of the first column. And is it an eigenvector? Yes, no? No. This vector is not in the same direction as that one. No good.

Now can you tell me one that is? You're going to guess it. [1, 1]. Try [1, 1]. Do the multiplication and what do you get? Right? If I input this vector y, what do I get out? Actually I get y itself. Right? The point is it didn't change direction, and it didn't even change length. So what's the eigenvalue for that? So I've got one eigenvalue now, one eigenvector. [1, 1]. And I've got the eigenvalue. So here are the vectors, the y's. And here are the lambdas. And I've got one of them and it's one, right? Would you like to guess the other one? I'm only looking for two because it's a two by two matrix. So let me erase here, hope that you'll come up with another one. [1, -1] is certainly worth a try. Let's test it. If it's an eigenvector, then it should come out in the same direction. What do I get when I do that? So I do that multiplication. Three and I get three and minus three, so have we got an eigenvector? Yep. And what's, so if this was y, what is this vector? 3y. So there's the other eigenvector, is [1, -1], and the other eigenvalue is three.

So we did it by spotting it here. MATLAB can't do it that way. It's got to figure it out. But we're ahead of MATLAB this time. So what do I notice? What do I notice about this matrix? It was symmetric. And what do I notice about the eigenvectors? If I show you those two vectors, [1, 1] and [1, -1], what do you see there? They're orthogonal. [1, 1] is orthogonal to [1, -1], perpendicular is the same as orthogonal. These are orthogonal, perpendicular. I can draw them, of course, and see that. [1, 1] will go, if this is one, it'll go here. So that's [1, 1]. And [1, -1] will go there, it'll go down, this would be the other one. [1, -1]. So there's y_1. There's y_2. And they are perpendicular. But of course I don't draw pictures all the time. What's the test for two vectors being orthogonal? The dot product. The dot product. The inner product. y transpose-- y_1 transpose * y_2. Do you prefer to write it as y_1 with a dot, y_2? This is maybe better because it's matrix notation. And the point is orthogonal, the dot product is zero. So that's good. Very good, in fact. So here's a very important fact. Symmetric matrices have orthogonal eigenvectors. What I'm trying to say is eigenvectors and eigenvalues are like a new way to look at a matrix. A new way to see into it. And when the matrix is symmetric, what we see is perpendicular eigenvectors.

And what comment do you have about the eigenvalues of this symmetric matrix? Remembering what was on the board for this anti-symmetric matrix. What was the point about that anti-symmetric matrix? Its eigenvalues were imaginary actually, an i there. Here it's the opposite. What's the property of the eigenvalues for a symmetric matrix that you would just guess? They're real. They're real. Symmetric matrices are great because they have real eigenvalues and they have perpendicular eigenvectors and actually, probably if a matrix has real eigenvalues and perpendicular eigenvectors, it's going to be symmetric. So symmetry is a great property and it shows up in a great way in this real eigenvalue, real lambdas, and orthogonal y's. Shows up perfectly in the eigenpicture.

Here's a handy little check on the eigenvalues to see if we got it right. Course we did. That's one and three we can get. But let me just show you two useful checks if you haven't seen eigenvalues before. If I add the eigenvalues, what do I get? Four. And I compare that with adding down the diagonal of the matrix. Two and two, four. And that check always works. The sum of the eigenvalues matches the sum down the diagonal. So that's like, if you got all the eigenvalues but one, that would tell you the last one. Because the sum of the eigenvalues matches the sum down the diagonal. You have no clue where that comes from but it's true.

And another useful fact. If I multiply the eigenvalues what do I get? Three? And now, where do you see a three over here? The determinant. 4-1=3. Can I just write those two facts with no idea of proof. The sum of the lambdas, I could write "sum". This is for any matrix, the sum of the lambdas is equal to the, it's called the trace, of the matrix. The trace of the matrix is the sum down the diagonal. And the product of the lambdas, lambda_1 times lambda_2 is the determinant of the matrix. Or if I had ten eigenvalues, I would multiply all ten and I'd get the determinant. So that's some facts about eigenvalues. There's more, of course, in section 1.5 about how you would find eigenvalues and how you would use them. That's of course the key point, is how would we use them.

Let me say something more about that, how to use eigenvalues. Suppose I have this system of 1,000 differential equations. Linear, constant coefficients, starts from some u(0). How do eigenvalues and eigenvectors help? Well, first I have to find them, that's the job. So suppose I find 1,000 eigenvalues and eigenvectors. A times eigenvector number i is eigenvalue number i times eigenvector number i. So these, y_1 to y_1000, so y_1 to y_1000 are the eigenvectors. And each one has its own eigenvalue, lambda_1 to lambda_1000. And now if I did that work, sort of like, in advance, now I come to the differential equation. How could I use this? This is now going to be the most-- it's three steps to use it, three steps to use these to get the answer. Ready for step one. Step one is break u nought into eigenvectors. Split, separate, write, express u(0) as a combination of eigenvectors.

Now step two. What happens to each eigenvector? So this is where the differential equation starts from. This is the initial condition. 1,000 components of u at the start and it's separated into 1,000 eigenvector pieces. Now step two is watch each piece separately. So step two will be multiply say c_1 by e^(lambda_1*t), by its growth. This is following eigenvector number one. And in general, I would multiply every one of the c's by e to those guys. So what would I have now? This is one piece of the start. And that gives me one piece of the finish. So the finish is, the answer is to add up the 1,000 pieces. And if you're with me, you see what those 1,000 pieces are. Here's a piece, some multiple of the first eigenvector. Now if we only were working with that piece, we follow it in time by multiplying it by e to the lambda_1 * t, and what do we have at a later time? c_1*e^(lambda_1*t)y_1. This piece has grown into that. And other pieces have grown into other things. And what about the last piece? So what is it that I have to add up? Tell me what to write here. c_1000, however much of eigenvector 1,000 was in there, and then finally, never written left-handed before, e to the who? Lambda number 1,000, not 1,000 itself, but its eigenvalue, 1,000, t. This is just splitting, this is constantly, constantly the method, the way to use eigenvalues and eigenvectors. Split the problem into the pieces that go-- that are eigenvectors. Watch each piece, add up the pieces. That's why eigenvectors are so important.

Yeah? Yes, right. Well, now, very good question. Let's see. Well, the first thing we have to know is that we do find 1,000 eigenvectors. And so my answer is going to be for symmetric matrices, everything always works. For symmetric matrices, if size is 1,000, they have 1,000 eigenvectors, and next time we'll have a shot at some of these. What some of them are for these special matrices. So this method always works if I've got a full family of independent eigenvectors. If it's of size 1,000, I need, you're right, exactly right. To see that this was the questionable step. If I haven't got 1,000 eigenvectors, I'm not going to be able to take that step. And it happens. I am sad to report that some matrices haven't got enough eigenvectors. Some matrices, they collapse. This always happens in math, somehow. Two eigenvectors collapse into one and the matrix is defective, like it's a loser.

So now you have to, of course, the equation still has a solution. So there has to be something there, but the pure eigenvector method is not going to make it on those special matrices. I could write down one but why should we give space to a loser? But what happens in that case? You might remember from differential equations when two of these roots, these are like roots, these lambdas are like roots that you found in solving a differential equation. When two of them come together, that's when the danger is. When I have a double eigenvalue, then there's a high risk that I've only got one eigenvector. And I'll just put in this little thing what the other, so the e^(lambda_1*t) is fine. But if that y_1 is like, if the lambda_1's in there twice, I need something new. And the new thing turns out to be t*e^(lambda* t). I don't know if anybody remembers. This was probably hammered back in differential equations that if you had repeated something or other then this, you didn't get pure e^(lambda*t)'s, you got also a t*e^(lambda*t). Anyway that's the answer. That if we're short eigenvectors, and it can happen, but it won't for our good matrices.

Okay, so Monday I've got lots to do. Special eigenvalues and vectors and then positive definite.