GRAPHICAL ANALYSIS OF ORBITS
&
FIXED POINTS OF FUNCTIONS

G. AUSTIN FORD

1. Introduction

This set of notes is a paraphrasing of Chapter 4 and Sections 1 through 4 of Chapter 5 in R. L. Devaney's *A First Course in Chaotic Dynamical Systems*. Some added material is included on the topic of fixed points from Chapter 5, specifically some proofs and discussion on the Banach and Brouwer Fixed Point Theorems.

2. Graphical Analysis of Orbits

In studying dynamical systems, one of the most important and enlightening things we can do is examine the behavior of the orbits of that system. This gives us an idea of how the system itself works. It is especially helpful to have a graphical representation of these orbits, for this allows us to get a more intuitive grasp on what is going on. There are two ways that we have of making such a graphical analysis: the orbit diagram and the phase portrait.

2.1. **Orbit Diagrams.** The first method we have of graphically plotting the behavior of an orbit is an orbit diagram. This is a two dimensional representation of an orbit, and it is incredibly useful when we are dealing with functions of one variable. To make an orbit diagram, we follow this short algorithm:

(1) Plot our given function $f(x)$ and the line $y = x$ in the plane.
(2) Pick an initial seed point x_0.
(3) Plot the point (x_0, x_0).
(4) Draw a vertical line from (x_0, x_0) to the point $(x_0, f(x_0))$.
(5) Draw a horizontal line from $(x_0, f(x_0))$ to the line $y = x$; call the point where these lines intersect (x_1, x_1).
(6) Return to step (3) with (x_1, x_1) as the new seed point.

This will product a staircase-like pattern that goes from the initial seed point through its orbit. Using this method, we can usually get an understanding of what's going on in a one-variable system. For a few examples, consult Figures 4.1 through 4.6 in Devaney's text.

2.2. **The Phase Portrait.** The second method we have of graphically studying the behavior of an orbit is a phase portrait. This is a one dimensional representation of an orbit, and it is our only method of studying systems of two variables or higher. To create a phase portrait, we follow an even shorter algorithm than that for an orbit diagram:

(1) Pick an initial seed point x_0.
(2) Plot x_0 on the real line.
(3) Compute the point $x_1 = f(x_0)$.
(4) Draw a look from x_0 to x_1.
(5) Return to step (2) with x_1 as our new point.

This will produce a horizontal line with a series of loops, each loop taking you from one point of the orbit to the next. This allows us to get at least a succinct representation of the orbit, which can often be quite illustrative. For a few examples, please take a look at Figures 4.7 and 4.8 of Devaney's text.

3. Fixed Points of Functions

In this section, we are concerned with a particular type of orbit of a function, namely fixed points. These are among the most important kinds of orbits a dynamical system can have, and it is important that we be able to determine where they are. We are also interested in examining the behavior of a function around these fixed points.

Let's begin with the definition of a fixed point.

Definition 1 (Fixed Point)**.** Let E be a subset of a metric space X with metric d, and let $f : E \longrightarrow X$ be a function. We say that $x \in E$ is a *fixed point* of f if $f(x) = x$.

Thus, fixed points are the points in the domain of a function that remain the same after the function has been applied. The following are some examples:

- $f(x) = 2x + 4$ and the point $x_0 = -4$
- $g(x) = x^2 - x + 1$ and the point $x_0 = 1$
- $h(x) = \sqrt{x/2 + 5}$ and the point $x_0 = 5/2$.

Now the following question arises: How do we determine whether or not a general function has a fixed point? It turns out that this has a many different answers. In the case of functions like the three above, we can determine the fixed points by simply setting the function equal to it's independent variable, i.e. setting $f(x) = x$, and solving. This, however, will not always work for a general function. There are many complicated functions for which such a calculation will become impossible, especially if we wish to find a solution explicitly. Ergo, we make use of some theoretical results which show the existence of fixed points for certain functions.

3.1. A Few Fixed Point Theorems.

We will look at three different examples of fixed point theorems: the Fixed Point Theorem, the Banach Fixed Point Theorem, and the Brouwer Fixed Point Theorem. These theorems each give a different type of function that has at least one fixed point. The first theorem is a result that follows from the basics of calculus, and the second only requires knowledge of Cauchy sequences and completeness. The Brouwer theorem, on the other hand, requires knowledge outside of 1-dimensional analysis, and it will only be stated.

3.1.1. *An Elementary Fixed Point Theorem.*

First of the theorems we will look at is the so-called Fixed Point Theorem. It relies on a well-known result from basic calculus, namely the Intermediate Value Theorem. To show that the Intermediate Value Theorem holds, we must first recall what it means for a space to be connected.

Definition 2 (Connected Metric Space and Separation)**.** A metric space (X, d) is *connected* if there does not exist a separation of X. By a *separation*, we mean a pair of disjoint open sets $A, B \subset X$ such that $X \subset A \cup B$ and $A \cap \bar{B} = \bar{A} \cap B = \varnothing$.

Therefore, we see that a space is connected if it cannot be divided up into two pieces which are at least some distance apart. Some examples of connected sets are the following:

- the real line $\mathbb{R}$
- the n-dimensional Euclidean space $\mathbb{R}^n$
- the n-dimensional complex space $\mathbb{C}^n$.

Let us now verify the Intermediate Value Theorem.

Theorem 1 (Intermediate Value Theorem)**.** *Let E be a subset of a connected metric space (X, d), and let $f : E \longrightarrow \mathbb{R}$ be a continuous function. Given points $x_1, x_2 \in E$ and a point y lying between $f(x_1)$ and $f(x_2)$, there is an $x \in E$ such that $f(x) = y$.*

Proof. We shall proceed by contradiction. Suppose that $f : E \longrightarrow \mathbb{R}$ is a continuous function, and suppose that for $x_1, x_2 \in E$ there is some $y \in \mathbb{R}$ lying between $f(x_1)$ and $f(x_2)$ such that $f(x) \neq y$ for any choice of $x \in E$. We then have that $f^{-1}((-\infty, y))$ and $f^{-1}((y, \infty))$ form a separation of E. However, we know that E is a connected set, which contradicts the existence of such a separation. Hence, we must have that there is some $x \in E$ such that $f(x) = y$. □

In particular, notice that functions $f : [a, b] \longrightarrow \mathbb{R}$ satisfy the Intermediate Value Theorem; this is because $[a, b]$ is a connected set. The proof of this is left as an exercise.

We now have all the tools we need to prove our first fixed point theorem.

Theorem 2 (Fixed Point Theorem)**.** *Let $f : [a, b] \longrightarrow [a, b]$ be a continuous function. Then f has a fixed point.*

Proof. Consider the function $F(x) = f(x) - x$. F is the sum of two continuous functions, so it is continuous. Since $a \leqslant f(x) \leqslant b$ for all $x \in [a, b]$, we have that

$$\begin{aligned} 0 \leqslant f(a) - a = F(a) \leqslant b - a \\ a - b \leqslant f(b) - b = F(b) \leqslant 0. \end{aligned} \tag{1}$$

Therefore, 0 lies between $F(a)$ and $F(b)$. By the Intermediate Value Theorem, we then have that there is some $x_0 \in [a, b]$ such that

$$F(x_0) = f(x_0) - x_0 = 0. \tag{2}$$

Thus, $f(x_0) = x_0$, so x_0 is a fixed point of the function f. □

Thus, we have that functions that carry a closed interval $[a, b]$ into itself always have some fixed point.

3.1.2. *The Banach Fixed Point Theorem.* The next fixed point theorem we will look at is the Banach Fixed Point Theorem. It requires a bit more sophistication than the previous theorem, namely knowledge of Cauchy sequences and complete metric spaces. Let us first remind ourselves what a sequence is and what it means for a sequence to converge.

Definition 3 (Sequences and Convergence)**.** Let (X, d) be a metric space. A *sequence* of points in X is a collection $\{x_n\}_{n=1}^{\infty}$ of points of X. Such a sequence is said to converge to $x \in X$ if given $\varepsilon > 0$, there is an $N > 0$ such that $d(x_n, x) < \varepsilon$ whenever $n \geqslant N$.

This tells us that a sequence converges to a point when, if you go far enough out into the sequence, all points come arbitrarily close to that point. Sequences are very important in analysis, and they arise in a myriad of situations. One particularly important example of a sequence is the Cauchy sequence.

Definition 4 (Cauchy Sequence)**.** A sequence $\{x_n\}_{n=0}^{\infty}$ in a metric space (X, d) is called a *Cauchy sequence* if given $\varepsilon > 0$ there is an $N > 0$ such that for all $m, n \geqslant N$ we have $d(x_m, x_n) < \varepsilon$.

We then see that a Cauchy sequence is a sequence that, under "nice" conditions, will always converge. This is because after enough time, the points in a Cauchy sequence will become arbitrarily close to one another; thus, if a Cauchy sequence does not converge in a space, it must be because the limiting point is simply not in the space. For instance, take the decimal expansion of $\sqrt{2}$. In the rational numbers, this is a Cauchy sequence; however, it does not converge inside of the rationals. However, if we consider this sequence in the reals, then it does indeed converge. This brings us to the notion of a complete metric space.

Definition 5 (Complete Metric Space)**.** A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges in X.

Now, we are ready to prove the Banach Fixed Point Theorem.

Theorem 3 (Banach Fixed Point Theorem)**.** *Let E be a subset of a nonempty, complete metric space (X, d), and let $f : E \longrightarrow X$ be a contraction, i.e. a map $f : X \longrightarrow X$ with constant $0 \leqslant \lambda < 1$ such that*

$$d(f(x), f(y)) \leqslant \lambda\, d(x, y) \tag{3}$$

for all $x, y \in X$. Then f has a unique fixed point x^*.

Proof. Let (X, d) be a nonempty complete metric space, and let f be a contraction mapping on (X, d) with constant λ. Pick an arbitrary $x_0 \in X$, and define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_n = f^n(x_0), \tag{4}$$

where f^n denotes the nth iteration of f. Let $a = d(f(x_0), x_0)$. We first show by induction that for any $n \geqslant 0$,

$$d(f^n(x_0), x_0) \leqslant \frac{1 - \lambda^n}{1 - \lambda} a. \tag{5}$$

For $n = 0$, this is clear. For any $n \geqslant 1$, suppose that $d(f^{n-1}(x_0), x_0) \leqslant \frac{1-\lambda^{n-1}}{1-\lambda}a$. Then

$$\begin{aligned} d(f^n(x_0), x_0) &\leqslant d(f^n(x_0), f^{n-1}(x_0)) + d(x_0, f^{n-1}(x_0)) \\ &\leqslant \lambda^{n-1} d(f(x_0), x_0) + \frac{1-\lambda^{n-1}}{1-\lambda}a \\ &= \frac{\lambda^{n-1} - \lambda^n}{1-\lambda}a + \frac{1-\lambda^{n-1}}{1-\lambda}a \\ &= \frac{1-\lambda^n}{1-\lambda}a \end{aligned} \tag{6}$$

by the triangle inequality and repeated application of the property $d(f(x), f(y)) \leqslant \lambda\, d(x, y)$ of f. By induction, the inequality holds for all $n \geqslant 0$. Given any $\varepsilon > 0$, it is possible to choose a positive integer N such that $\frac{\lambda^n}{1-\lambda}a < \varepsilon$ for all $n \geqslant N$, because $\frac{\lambda^n}{1-\lambda}a \to 0$ as $n \to \infty$. Now, for any $m, n \geqslant N$ (we may assume that $m \geqslant n$),

$$\begin{aligned} d(x_m, x_n) &= d(f^m(x_0), f^n(x_0)) \\ &\leqslant \lambda^n d(f^{m-n}(x_0), x_0) \\ &\leqslant \lambda^n \frac{1-\lambda^{m-n}}{1-\lambda}a \\ &< \frac{\lambda^n}{1-\lambda}a \\ &< \varepsilon, \end{aligned} \tag{7}$$

so the sequence $\{x_n\}$ is a Cauchy sequence. Because (X, d) is complete, this implies that the sequence has a limit in (X, d); define x^* to be this limit. We now prove that x^* is a fixed point of f. Suppose it is not; then $\delta = d(f(x^*), x^*) > 0$. However, because $\{x_n\}$ converges to x^*, there is a positive integer N such that $d(x_n, x^*) < \delta/2$ for all $n \geqslant N$. Then

$$\begin{aligned} d(f(x^*), x^*) &\leqslant d(f(x^*), x_{N+1}) + d(x^*, x_{N+1}) \\ &\leqslant \lambda\, d(x^*, x_N) + d(x^*, x_{N+1}) \\ &< \delta/2 + \delta/2 = \delta, \end{aligned} \tag{8}$$

which is a contradiction. So x^* is a fixed point of f. It is also unique. Suppose there is another fixed point x' of T; because $x' \neq x^*$, $d(x', x^*) > 0$. But then

$$d(x', x^*) = d(f(x'), f(x^*)) \leqslant \lambda\, d(x', x^*) < d(x', x^*), \tag{9}$$

a contradiction. Therefore, x^* is the unique fixed point of f. □

Hence, we have that contractions in complete metric spaces have fixed points. This is an especially important result of analysis, for it is used in the proof of Picard's Existence Theorem for solutions to initial value problems for ordinary differential equations.

3.1.3. *The Brouwer Fixed Point Theorem.* Lastly, we come to the Brouwer Fixed Point Theorem. This is the most advanced of the fixed points theorems we will look at. It involves self-maps of the unit ball $\mathbb{B}^n$ in n-dimensional Euclidean space. Let us first define what the unit n-ball is.

Definition 6 (The n-dimensional Unit Ball)**.** The *n-dimensional unit ball* $\mathbb{B}^n$ is the subset of $\mathbb{R}^n$ given by

$$\mathbb{B}^n := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : \|\mathbf{x}\| = [x_1^2 + \cdots + x_n^2]^{1/2} \leqslant 1\}. \tag{10}$$

Now we state the theorem.

Theorem 4 (Brouwer Fixed Point Theorem)**.** *If $f : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ is a continuous mapping, then f has a fixed point.*

Proof. This proof requires knowledge beyond 1-dimensional rigorous calculus, so it is omitted. Please consult either Jim Munkres' *Topology*, Victor Guillemin's "Supplementary Notes" to 18.101, or a similar text for a proof. □

This theorem is also an important one in analysis. For instance, it is used to prove various existence results in the field of partial differential equations, and it has many other applications as well.

3.2. **Attraction and Repulsion.** We now wish to examine how functions behave around their fixed points. Do functions send the nearby points towards a given fixed point? What about away from it? Is it possible that the function does none of the above?

In order to find answers to these questions, we return to the tools we obtained from calculus. To illustrate, let us consider the function $f(x) = x^2$. It has fixed points at $x_1 = 0$ and $x^2 = 1$. If we look at the behavior of the points in the interval $(0, 1)$ as f is applied, we see that they grow smaller and smaller as the number of iterations increases. Thus, we see that if $x \in (0, 1)$, then $f^n(x) \to 0$ as $n \to \infty$. Similarly, if we look at points x in the interval $(1, \infty)$, then as $f^n(x) \to \infty$ as $n \to \infty$. Lastly, the case for negative values of x is easily reduced to the symmetric positive case. This suggests that f sends points around 0 towards 0 and points around 1 away from 1. We would then call 0 an attracting fixed point, and we would call 1 a repelling fixed point. In general, these are defined as follows.

Definition 7 (Attracting, Repelling, and Neutral Fixed Points)**.** Let $E \subset \mathbb{R}$, and $f : E \longrightarrow \mathbb{R}$ be a function with fixed point $x_0 \in X$. Suppose f is continuously differentiable at x_0. We say that x_0 is an *attracting fixed point* if $|f'(x_0)| < 1$, and we say that x_0 is a *repelling fixed point* if $|f'(x_0)| > 1$. Lastly, if $|f'(x_0)| = 1$, we call x_0 a *neutral fixed point.*

In terms of graphical analysis, these definitions make intuitive sense. This is because the value of $f'(x_0)$ is the slope of the tangent line to the function's graph at x_0. If this has absolute value smaller than 1, then the tangent line will be below the line $y = x$ for $x > x_0$, and the tangent line will be above $y = x$ for $x < x_0$. This will cause the orbit diagram to move inward towards (x_0, x_0). Analogously, if this tangent line has positive slope at x_0, then it will be above $y = x$ for $x > x_0$, and it will be below $y = x$ for $x < x_0$. This will cause the orbit diagram to move outwards from (x_0, x_0). In the event that the derivative is ± 1 at x_0, this method becomes inconclusive. We must then rely on graphical analysis to determine the behavior of the function.

3.2.1. *Attracting Fixed Points.* The first type of fixed point that we mentioned was the attracting fixed point. Around this type of fixed point, orbits move inward. Some examples of functions and attracting fixed points are the following:

- $f(x) = -2x^2 + 2x$ and the point $x_0 = 1/2$
- $g(x) = x/2 + 6$ and the point $x_0 = 12$
- $h(x) = \sqrt{x}$ and the point $x_0 = 1$.

One might naturally wonder why this kind of behavior occurs, that is to say why attracting fixed points cause orbits to sink into them. The following theorem offers an answer.

Theorem 5 (Attracting Fixed Point Theorem)**.** *Let $E \subset \mathbb{R}$, and let $f : E \longrightarrow \mathbb{R}$ be a function with attracting fixed point x_0. Then there is an $\varepsilon > 0$ such that if $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, then $f^n(x) \in (x_0 - \varepsilon, x_0 + \varepsilon)$ for all positive n, and moreover $f^n(x) \to x_0$ as $n \to \infty$.*

Proof. Let everything be as in the statement of the theorem. We then have that since $|f'(x_0)| < 1$, there is a number $\lambda > 0$ such that $|f'(x_0)| < \lambda < 1$. By continuity of f', we may then choose an $\varepsilon > 0$ such that $|f'(x)| < \lambda$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Consider now an arbitrary $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. From the mean value theorem, we have that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(y) \tag{11}$$

for some y lying between x and x_0; thus, we have that

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} = |f'(y)| < \lambda, \tag{12}$$

or equivalently,

$$|f(x) - f(x_0)| < \lambda |x - x_0|. \tag{13}$$

We know that x_0 is a fixed point, so this gives that

$$|f(x) - x_0| < \lambda |x - x_0|. \tag{14}$$

Therefore, the distance from $f(x)$ to x_0 is smaller than the distance from x to x_0 since $0 < \lambda < 1$. This implies that $f(x_0)$ lies in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ since x was chosen in that interval. We can thus apply the above process to $f(x)$ and $f(x_0)$, giving

$$|f^2(x) - x_0| = |f^2(x) - f^2(x_0)| < \lambda |f(x) - f(x_0)| < \lambda^2 |x - x_0|. \tag{15}$$

Since $0 < \lambda < 1$, we have that $0 < \lambda^2 < \lambda$. This shows that as we continue to apply f to x and x_0, we shrink the distance between $f^n(x)$ and $f^n(x_0) = x_0$. This yields that for $n > 0$ we have

$$|f^n(x) - x_0| < \lambda^n |x - x_0|. \tag{16}$$

Note that $\lambda^n \to 0$ as $n \to \infty$. Hence, we have that $f^n(x) \to x_0$ as $n \to \infty$. □

We then see that under application of f, attracting fixed points actually do bring the nearby points towards them. In practice, this type of fixed point is much easier to find than the repelling fixed point. This is because when we compute the orbit of a given seed, we know that if it enters one of these critical intervals $(x_0 - \varepsilon, x_0 + \varepsilon)$ around an attracting fixed point x_0, it won't be able to escape.

3.2.2. *Repelling Fixed Points.* The second type of fixed point that we mentioned was the repelling fixed point; these are fixed points that cause orbits to move outward from them. Some examples of functions with such fixed points are:

- $f(x) = -2x^2 + 2$ and the point $x_0 = 0$
- $g(x) = 2x + 3$ and the point $x_0 = -3$
- $h(x) = x^4/4 - 5x + 8$ and the point $x_0 = 2$.

As with the case of the attracting fixed point, it is natural to wonder why this behavior appears. The following theorem, which is an analogue of the theorem for attracting fixed points, offers an answer.

Theorem 6 (Repelling Fixed Point Theorem)**.** *Let $E \subset \mathbb{R}$, and let $f : E \longrightarrow \mathbb{R}$ be a function with repelling fixed point x_0. Then there is an $\varepsilon > 0$ such that if $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ and $x \neq x_0$, then there is a positive integer N such that $f^N(x) \notin (x_0 - \varepsilon, x_0 + \varepsilon)$.*

Proof. Let everything be as in the statement of the theorem. Since $|f'(x_0)| > 1$, we have that there is some $\lambda > 0$ such that $|f'(x_0)| > \lambda > 1$. Furthermore, by continuity of f' we may choose a $\varepsilon > 0$ such that if $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, then $|f'(x)| > \lambda$. Choose now an arbitrary $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. From the mean value theorem, we have that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(y_1) \tag{17}$$

for some y_1 lying between x and x_0; therefore, we have that

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} = |f'(y_1)| > \lambda, \tag{18}$$

or equivalently,

$$|f(x) - f(x_0)| > \lambda |x - x_0|. \tag{19}$$

We know by assumption that x_0 is a fixed point of f; this gives that

$$|f(x) - x_0| > \lambda |x - x_0|. \tag{20}$$

Since $\lambda > 1$, we have that $f(x)$ is further from x_0 than x originaly was. If $f(x) \notin (x_0 - \varepsilon, x_0 + \varepsilon)$, i.e. if $|f(x) - x| > \varepsilon$, then we are finished. If this is not the case, then we may repeat the above process using instead $f(x)$ as our point in $(x_0 - \varepsilon, x_0 + \varepsilon)$. This gives that

$$|f^2(x) - x_0| = |f^2(x) - f^2(x_0)| > \lambda |f(x) - f(x_0)| > \lambda^2 |x - x_0|. \tag{21}$$

Note that $\lambda^2 > \lambda$ since $\lambda > 1$. We then have that as we continue to apply f to x and x_0, we increase the distance between $f^n(x)$ and $f^n(x_0) = x_0$. This yields that for $n > 0$, we have

$$|f^n(x) - x_0| > \lambda^n |x - x_0| \tag{22}$$

as long as $f^{n-1}(x) \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Note that $\lambda^n \to \infty$ as $n \to \infty$. We then have that there is some $N > 0$ such that $\lambda^n > \varepsilon/|x - x_0|$ for $n \geqslant N$. This tells us that

$$|f^N(x) - x_0| > \lambda^N |x - x_0| > \varepsilon. \tag{23}$$

Hence, $f^N(x) \notin (x_0 - \varepsilon, x_0 + \varepsilon)$. □

From the proof of this theorem, we are able to see that repelling fixed points literally push the points around them away the more we apply f. This makes this type of fixed point rather difficult to find when one does analysis of a dynamical system, for the points within a critical interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ do not remain in that interval for long once we begin iterating f.

3.2.3. *Neutral Fixed Points.* The last type of fixed point to discuss is the neutral fixed point. These are the fixed points around which it is impossible to discover the behavior of the dynamical system by using only analytical methods. We must instead revert to our tools of numerical calculation and graphical analysis. These will allow us to pinpoint which side of such a neutral fixed point serves to attract the nearby points to it, and which side of a neutral fixed point serves to repel the nearby points away from it. Some examples of functions with neutral fixed points are the following:

- $f(x) = x - x^2$ and the point $x_0 = 0$
- $g(x) = 1/x$ and the point $x_0 = 1$
- $h(x) = x$ and any point $x_0 \in \mathbb{R}$.