Lecture Notes: Chapter 11
Sarkovskii's Theorem

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1 Period 3 Implies Chaos

Theorem 1 (The Period 3 Theorem)**.** *Suppose $F : \Re \to \Re$ is continuous. Suppose also that F has a periodic point of prime period 3. Then F also has periodic points of all other periods.*

Useful Observations The following statements will be helpful in proving Theorem 1. Pictorial demonstrations of both are in the textbook on page 135, figures 11.1 and 11.2.

Observation 1. *Suppose $I = [a, b]$ and $J = [c, d]$ are closed intervals and $I \subset J$. If $F(I) \supset J$, then F has a fixed point in I.*

This follows immediately from the intermediate value theorem. Since $I \subset J$, the graph of F must cross the diagonal. The fixed point, of course, does not need to be unique.

Observation 2. *Suppose I and J are two closed intervals and $F(I) \supset J$. Then there is a closed subinterval $I' \subset I$ such that $F(I') = J$.*

Note that we do not assume that $I \subset J$ in this case.

Proof

Suppose that the 3-cycle of F is given by

$$a \mapsto b \mapsto c \mapsto a \mapsto \dots \tag{1}$$

Assume that a is the leftmost point on the orbit. There are two possibilities then for the relative positions of b and c. We will assume that $a < b < c$. The other case is proven similarly.

Let $I_0 = [a, b]$ and $I_1 = [b, c]$. Since $F(a) = b$, $F(b) = c$, and F is continuous, by the Intermediate Value Theorem

$$F(I_0) \supset I_0. \tag{2}$$

Similarly, since $F(b) = c$ and $F(c) = a$,

$$F(I_1) \supset I_0 \cup I_1. \tag{3}$$

We will next construct cycles of length 1 and 2. Then we will construct all cycles of length $n > 3$.

N=1 Since $F(I_1) \supset I_1$ (by (3)), there is a fixed point in I_1 (Observation 1).

N=2 Since $F(I_0) \supset I_1$ (by (2)) and $F(I_1) \supset I_0$ (by (3)), $F^2(I_0) \supset I_0$. So there is a fixed point of F^2 in I_0 (Observation 1). So F has a 2-cycle.

Cycles of Length Greater Than 3

To find a periodic point with period n, we will invoke Observation 2 a total of n times.

Since $F(I_1) \supset I_1$, there is a closed subinterval $A_1 \subset I_1$ such that $F(A_1) = I_1$.

Again invoking Observation 2, since $F(A_1) \supset A_1$, we can find a closed subinterval $A_2 \subset A_1$ such that $F(A_2) = A_1$. Note that by construction, $A_2 \subset A_1 \subset I_1$.

Repeat this process $n-2$ times. We this end up with a collection of closed subintervals

$$A_{n-2} \subset A_{n-3} \subset \ldots \subset A_2 \subset A_1 \subset I_1 \tag{4}$$

Note that $F^{n-2}(A_{n-2}) = I_1$ and $A_{n-2} \subset I_1$.

Since $F(I_0) \supset I_1 \supset A_{n-2})$, there is a closed subinterval $A_{n-1} \subset I_0$ such that $F(A_{n-1}) = A_{n-2}$.

Lastly, since $F(I_1) \supset I_0 \supset A_{n-1}$, there is a closed subinterval $A_n \subset I_1$ such that $F(A_n) = A_{n-1}$.

We have now constructed a series of closed intervals such that

$$A_n \mapsto A_{n-1} \mapsto A_{n-2} \ldots \mapsto A_2 \mapsto A_1 \mapsto I_1. \tag{5}$$

Since $F^n(A_n) = I_1$ and $A_n \subset I_1$, we may invoke Observation 1 to conclude that there is a point, x_0 fixed by F^n.

We must now show that the orbit of x_0 has prime period n. Note that $x_0 \notin I_0 \cap I_1$, since $I_0 \cap I_1 = \{b\}$ and $F(b) = c \notin I_0$, whereas $F(x_0) \in F(A_n \subset I_0$.

$F(x_0) \in I_0$, but all other iterations lie in I_1. So x_0 cannot have period less than n.

This completes the proof.

2 Sarkovskii's Theorem

2.1 The Sarkovskii Ordering of the Natural Numbers

The following ordering, read from left-to-right, then top-to-bottom, is known as Sarkovskii's Ordering of the Natural Numbers.

$$\begin{aligned}
&3, 5, 7, 9, \ldots \\
&2 \cdot 3, 2 \cdot 5, 2 \cdot 7, \ldots \\
&2^2 \cdot 3, 2^2 \cdot 5, 2^2 \cdot 7, \ldots \\
&2^3 \cdot 3, 2^3 \cdot 5, 2^3 \cdot 7, \ldots \\
&\vdots \\
&\ldots, 2^n, \ldots, 2^3, 2^2, 2, 1
\end{aligned}$$

2.2 Sarkovskii's Theorem

Theorem 2 (Sarkovskii's Theorem)**.** *Suppose* $F : \Re \to \Re$ *is continuous. Suppose that* F *has a periodic point of period* n *and that* n *precedes* k *in the Sarkovskii ordering. Then* F *also has a periodic point of prime period* k*.*

The proof is very similar to the proof for $n = 3$ which we did above. The converse (which is stated here without proof) turns out to be also true:

Theorem 3. *There is a continuous function* $F : \Re \to \Re$ *which has a cycle of period* n*, but no cycles of any period that precedes* n *in the Sarkovskii ordering.*

2.3 Comments about Sarkovskii's Theorem

1. Since the number 2^n form the tail of the ordering, any function that only has a finite number of cycles will have all cycles with period equal to a power of 2. This is part of why we see period doubling as a family of functions transitions to chaos.

2. The theorem only applies to the real number line. For example, the function defined on a circle that rotates all points by a fixed angle $2\pi/n$ has periodic points of period n but no others.

3. The infinity of other cycles doesn't appear on the orbit diagram of $Q_\lambda(x)$ because the others are repelling cycles