

The Schwarzian Derivative & the Critical Orbit

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1 Schwarzian Derivative

What we introduce here is a very strange looking derivative, but it turns out to give some useful information about the behavior of many functions, particularly at their critical points.

Definition: The **Schwarzian derivative** (SD for short) of f is

$$\mathcal{S}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2.$$

Devaney's text has a few example of functions that we have been examining which have negative SD, and in those cases the SD at the critical points is $-\infty$. As it will turn out, this property of having a negative SD is important.

Proposition: Let $p(x)$ be a polynomial and suppose all of the roots of $p'(x)$ are distinct and real. Then $\mathcal{S}p < 0$.

Proof: Since all of the roots of p' are distinct, we can factor p' as $p'(x) = \alpha(x - \beta_1) \cdots (x - \beta_n)$, where β_1, \dots, β_n are the n distinct roots of p' , with each $\beta_i \in \mathbb{R}$. Taking the logarithm of both sides of this equation allows us to split this product up into a summation. Differentiating kills off the constant, and dividing through by p' gives:

$$\frac{p''(x)}{p'(x)} = \sum_{i=1}^n \frac{1}{x - \beta_i}.$$

Using the quotient rule to differentiate again, we see:

$$\begin{aligned} \frac{p'''(x)p'(x) - (p''(x))^2}{(p'(x))^2} &= \frac{p'''(x)}{p'(x)} - \left(\frac{p''(x)}{p'(x)} \right)^2 \\ &= - \sum_{i=1}^n \frac{1}{(x - \beta_i)^2}. \end{aligned}$$

This gives

$$\mathcal{S}p(x) = - \sum_{i=1}^n \frac{1}{(x - \beta_i)^2} - \frac{1}{2} \left(\sum_{i=1}^n \frac{1}{x - \beta_i} \right)^2 < 0.$$

Chain Rule for Schwarzian Derivatives: Let f and g be two functions, then

$$\mathcal{S}(f \circ g)(x) + \mathcal{S}f(g(x)) \cdot (g'(x))^2 + \mathcal{S}g(x).$$

Proof: Omitted, see the text.

Corollary: Suppose $\mathcal{S}f < 0$ and $\mathcal{S}g < 0$, then $\mathcal{S}(f \circ g) < 0$. Along these liens, if $\mathcal{S}f < 0$, then $\mathcal{S}f^n < 0$.

Proof: This is a direct consequence of the chain rule for the SD, since $\mathcal{S}f(g(x)) < 0$ and $\mathcal{S}g(x) < 0 \forall x$.

Here is one of the more interesting applications of the SD:

Schwarzian Max-Min Principle: Suppose $\mathcal{S}f < 0$, then f' cannot have a positive local minimum or a negative local maximum.

Proof: Let x_0 be a critical point of f' . This means that $f''(x_0) = 0$. Suppose that $f'(x_0) \neq 0$, then

$$\mathcal{S}f(x_0) = \frac{f'''(x_0)}{f'(x_0)} < 0.$$

Now suppose f' has a positive local minimum at x_0 , then by elementary calculus, $f'''(x_0) \geq 0$, but if x_0 is a local minimum then $f'(x_0) > 0$, which would cause $\mathcal{S}f(x_0) \geq 0$, contradiction. Considering a negative local maximum at x_0 gives a similar contradiction.

Figure 12.1 in the text offers types of graphs which cannot occur due to this Max-Min Principle.

2 Critical Point and Basins of Attraction

Now we will use the SD to actually analyze the behavior of some functions. First, we need to define some things.

Definition: If x_0 is an attracting fixed point for a function f , the **basin of attraction** of x_0 is of the set of all points whose orbits tend to x_0 . The **immediate basin of attraction** is the largest interval containing x_0 that lies in the basin of attraction.

It is usually the case that the immediate basin is smaller than the full basin, but it is possible for them to be the same. If $f(x) = x^2$, for example, $(-1, 1)$ is the basin of attraction of 0, and it is also the immediate basin. In terms of the SD, we can now say the following:

Theorem: Suppose f has negative SD. If x_0 is an attracting periodic point of f , then either the immediate basin of attraction of x_0 extends to $+\infty$ or $-\infty$, or else there is a critical point of f whose orbit is attracted to the orbit of x_0 .

Proof: Omitted, though the text offers a proof of one of the simple cases of the theorem.

As an immediate application of this theorem to the quadratic map Q_c , we note that if $|x|$ is large enough, then the orbit tends to $\pm\infty$. Thus, the second part of the theorem applies and we conclude that Q_c has at most one attracting cycle since Q_c has only one critical point, and $\mathcal{S}Q_c < 0$.

This theorem helps explain why critical points are used to plot orbit diagrams, since the critical point will “find” the attracting cycles of the function.

Example: Consider $A_\lambda(x) = \lambda \arctan x$. We have then that

$$A'_\lambda(x) = \frac{\lambda}{1+x^2}.$$

If $\lambda \neq 0$, then $A'_\lambda(x) \neq 0$, so A_λ has no critical points. If $|\lambda| < 1$, then 0 is an attracting fixed point, with the immediate basin of attraction being the whole real line. If $\lambda > 1$, then A_λ has two attracting fixed points, and the basins extend to infinity. Along the same lines, when $\lambda < -1$, A_λ has an attracting 2-cycle, and the basins extend to infinity.