Newton's Method

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1 Construction

Suppose we want to find a root of F, that is solution of $F(x) = 0$. For most functions we can't algebraically solve the equations and must use numerical techniques. One method for doing this, that you may have seen in calculus, is the Newton-Raphson method. Given an initial guess, x_0, draw the tangent to the graph of F at $(x_0, F(x_0))$. Unless we have had the bad luck of picking a critical point of F, this line intersects the x-axis at a new point, x_1, this point is out new guess. Algebraically we get

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}.$$

Newton's method consists of iterating this procedure. Hence we define the *Newton iteration function* associated to F to be

$$N(x) = x - \frac{F(x)}{F'(x)}.$$

2 Convergence

We must first define the multiplicity of a root.

Definition. A root x_0 of F has *multiplicity* k, if $F^{[k-1]}(x_0) = 0$, but $F^{[k]}(x_0) \neq 0$. Here $F^{[k]}$ is the k^{th} derivative of F.

If x_0 is a root of F with multiplicity k, F can be written in the form $F(x) = (x - x_0)^k G(x)$ where G doesn't have a root at x_0. Note, however, that the multiplicity of a root can be infinite.

Newton's Fixed Point Theorem. Suppose F is a (sufficiently differentiable) function and N is its associated Newton iteration function. Then, assuming all roots of F have finite multiplicity, x_0 is a root of multiplicity k if and only if x_0 is a fixed point of N. Moreover, such a fixed point is always attracting.

Proof. Suppose first that x_0 has multiplicity 1, i.e., $F(x_0) = 0$, but $F'(x_0) \neq 0$. Then it is clear that $N(x_0) = x_0$. Conversely, $N(x_0) = x_0$ implies $F(x_0) = 0$. Next, we compute

$$N'(x) = \frac{F(x)F''(x)}{(F'(x))^2}$$

using the quotient rule. Hence if x_0 has multiplicity 1, $N'(x_0) = 0$ so x_0 is indeed attracting.

For the general case, see text. $\square$

Despite the above theorem, Newton's method doesn't always converge. One problem is that F might not be differentiable. For example, if $F(x) = x^{1/3}$, then $N(x) = -2x$ which has a repelling fixed point at 0, the root of F.

Even if F is differentiable, there may still be problems with cycles. Let $F(x) = x^3 - 5x$. Then we see

$$N(x) = x - \frac{x^3 - 5x}{3x^2 - 5}.$$

This has a cycle since $N(1) = -1$ and $N(-1) = 1$. Therefore if we had made the initial guess $x_0 = 1$, Newton's method would have gotten stuck. In this case the cycle is repelling and so most initial guesses converge to a root.

This is not always the case. Consider $F(x) = (x^2 - 1)(x^2 + A)$. From the proof of Newton's fixed point theorem, N has critical points at the places where F'' vanishes. Hence the points

$$c_\pm = \pm\sqrt{\frac{1 - A}{6}}$$

are critical for N. If we set $A = (29 - \sqrt{720})/11$, the points $c_\pm$ lie on a 2-cycle, which is therefore attracting.