ℓ^p IS COMPLETE

Let $1 \leq p \leq \infty$, and recall the definition of the metric space ℓ^p:

$$\text{For } 1 \leq p < \infty,\ \ell^p = \left\{ \text{sequences } \mathbf{a} = (a_n)_{n=1}^\infty \text{ in } \mathbb{R} \text{ such that } \sum_{n=1}^\infty |a_n|^p < \infty \right\};$$

whereas ℓ^∞ consists of all those sequences $\mathbf{a} = (a_n)_{n=1}^\infty$ such that $\sup_{n\in\mathbb{N}} |a_n| < \infty$. We defined the p-norm as the function $\|\cdot\|_p \colon \ell^p \to [0,\infty)$, given by

$$\|\mathbf{a}\|_p = \left(\sum_{n=1}^\infty |a_n|^p \right)^{1/p}, \text{ for } 1 \leq p < \infty,$$

and $\|\mathbf{a}\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$. In class, we showed that the function $d_p \colon \ell^p \times \ell^p \to [0,\infty)$ given by $d_p(\mathbf{a},\mathbf{b}) = \|\mathbf{a}-\mathbf{b}\|_p$ is actually a metric. We now proceed to show that (ℓ^p, d_p) is a *complete* metric space for $1 \leq p \leq \infty$. For convenience, we will work with the case $p < \infty$, as the case $p = \infty$ requires slightly different language (although the same ideas apply).

Suppose that $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \ldots$ is a Cauchy sequence in ℓ^p. Note, each term $\mathbf{a}^k$ in the sequence is a point in ℓ^p, and so is itself a sequence:

$$\mathbf{a}^k = (a_1^k, a_2^k, a_3^k, \ldots).$$

Now, to say that $(\mathbf{a}^k)_{k=1}^\infty$ is a Cauchy sequence in ℓ^p is precisely to say that

$$\forall \epsilon > 0\, \exists K \in \mathbb{N} \text{ s.t. } \forall k, m \geq K,\ \|\mathbf{a}^k - \mathbf{a}^m\|_p < \epsilon.$$

That is, for given $\epsilon > 0$ and sufficiently large k, m, we have

$$\sum_{n=1}^\infty |a_n^k - a_n^m|^p = \|\mathbf{a}^k - \mathbf{a}^m\|_p^p < \epsilon^p.$$

Now, the above series has all non-negative terms, and hence is an upper bound for any *fixed* term in the series. That is to say, for fixed $n_0 \in \mathbb{N}$,

$$|a_{n_0}^k - a_{n_0}^m| \leq \sum_{n=1}^\infty |a_n^k - a_n^m|^p < \epsilon^p,$$

and so we see that the sequence $(a_{n_0}^k)_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. But we know that $\mathbb{R}$ is a complete metric space, and thus there is a limit $a_{n_0} \in \mathbb{R}$ to this sequence. This holds for each $n_0 \in \mathbb{N}$. The following diagram illustrates what's going on.

$$\begin{array}{ccccccc}
\mathbf{a}^1 & = & a_1^1 & a_2^1 & a_3^1 & a_4^1 & \cdots \\
\mathbf{a}^2 & = & a_1^2 & a_2^2 & a_3^2 & a_4^2 & \cdots \\
\mathbf{a}^3 & = & a_1^3 & a_2^3 & a_3^3 & a_4^3 & \cdots \\
\mathbf{a}^4 & = & a_1^4 & a_2^4 & a_3^4 & a_4^4 & \cdots \\
 & & \vdots & \vdots & \vdots & \vdots & \ddots \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & & a_1 & a_2 & a_3 & a_4 & \cdots
\end{array}$$

So, we have shown that, in this ℓ^p-Cauchy sequence of horizontal sequences, each *vertical* sequence actually converges. Hence, there is a sequence $\mathbf{a} = (a_1, a_2, a_3, a_4, \ldots)$ to which "$\mathbf{a}^k$ converges" in a vague sense. The sense is the "point-wise convergence" along vertical lines in the above diagram. To be more precise, recall that a sequence $\mathbf{a}$ is a function $\mathbf{a}\colon \mathbb{N} \to \mathbb{R}$, where we customarily write $\mathbf{a}(n) = a_n$. What we have shown is that, if $(\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \ldots)$ is a Cauchy sequence of such ℓ^p functions, then there is a function $\mathbf{a}\colon \mathbb{N} \to \mathbb{R}$ such that $\mathbf{a}^k$ converges to $\mathbf{a}$ *point-wise*; i.e. $\mathbf{a}^k(n) \to \mathbf{a}(n)$ for each $n \in \mathbb{N}$.

Now, our goal is to find a point $\mathbf{b} \in \ell^p$ such that $\mathbf{a}^k \to \mathbf{b}$ as $k \to \infty$ in the sense of ℓ^p; that is, such that $\|\mathbf{a}^k - \mathbf{b}\|_p \to 0$ as $k \to \infty$. The putative choice for this $\mathbf{b}$ is the sequence $\mathbf{a}$ given above. In order to show that one works, we need to show first that it is actually an ℓ^p sequence, and second that $\mathbf{a}^k$ converges to $\mathbf{a}$ in ℓ^p sense, not just point-wise.

To do this, it is convenient to first pass to a family of subsequences of the (a^k_n), as follows. Since $(a^k_1)_{k=1}^\infty$ converges to a_1, we can choose k_1 so that for $k \ge k_1$, $|a_1^{k_1} - a_1| < \frac{1}{2}$. Having done so, and knowing that $a_2^k \to a_2$, we can choose a larger k_2 so that for $k \ge k_2$, we have $|a_1^k - a_1| < \frac{1}{4}$ and $|a_2^k - a_2| < \frac{1}{4}$. Continuing this way iteratively, we can find an increasing sequence of integers $k_1 < k_2 < k_3 < \cdots$ such that

$$\text{for each } j \in \mathbb{N},\ |a_n^k - a_n| < 2^{-j} \text{ for } n = 1, 2, \ldots, j \text{ and } k \ge k_j. \tag{1}$$

In particular, we have $|a_n^{k_j} - a_n| < 2^{-j}$ for $j \ge n$. That gives us the following.

Lemma 1. *The sequence $\mathbf{a} = (a_n)_{n=1}^\infty$ of point-wise limits of $(\mathbf{a}^k)_{k=1}^\infty$ is in ℓ^p.*

Proof. Fix $N \in \mathbb{N}$, and recall that the finite-dimensional versions of the ℓ^p-norms,

$$\|(a_1, \ldots, a_N)\|_p = \left(\sum_{n=1}^N |a_n|^p\right)^{1/p}$$

also satisfy the triangle inequality (i.e. $d_p(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_p$ is a metric on $\mathbb{R}^N$). Hence, we can estimate the initial-segment of N terms of $\mathbf{a}$ as follows:

$$a_n = (a_n - a_n^{k_N}) + a_n^{k_N},$$

and so

$$\left(\sum_{n=1}^N |a_n|^p\right)^{1/p} \le \left(\sum_{n=1}^N |a_n - a_n^{k_N}|^p\right)^{1/p} + \left(\sum_{n=1}^N |a_n^{k_N}|^p\right)^{1/p}. \tag{2}$$

Now, the last term in Equation 2 is bounded by the actual ℓ^p-norm of the whole sequence $\mathbf{a}^{k_N}$; that is, we can tack on the infinitely many more terms,

$$\left(\sum_{n=1}^N |a_n^{k_N}|^p\right)^{1/p} \le \left(\sum_{n=1}^\infty |a_n^{k_N}|^p\right)^{1/p} = \|\mathbf{a}^{k_N}\|_p.$$

Recall that $(\mathbf{a}^k)_{k=1}^\infty$ is a Cauchy sequence in the metric space ℓ^p. We have proved that any Cauchy sequence in a metric space is *bounded*. Thus, there is a constant R independent of N such that $\|\mathbf{a}^{k_N}\|_p \le R$. Combining this with Equation 1, we can therefore estimate the right-hand-side of Equation 2 by

$$\left(\sum_{n=1}^N |a_n|^p\right)^{1/p} \le \left(\sum_{n=1}^N (2^{-N})^p\right)^{1/p} + R = \left(N\, 2^{-Np}\right)^{1/p} + R.$$

Finally, the term $\left(N\,2^{-Np}\right)^{1/p} = N^{1/p}2^{-N}$ converges to 0 as $N \to \infty$ (remember your calculus!), and hence this sequence is also bounded by some constant S. In total, then, we have

$$\left(\sum_{n=1}^{N} |a_n|^p\right)^{1/p} \leq R + S \text{ for all } N \in \mathbb{N}.$$

In other words, $\sum_{n=1}^{N} |a_n|^p \leq (R+S)^p$. The constant on the right does not depend on N; it is an upper bound for the increasing sequence of partial sums of the series $\sum_{n=1}^{\infty} |a_n|^p = \|\mathbf{a}\|_p^p$. Thus, we have $\|\mathbf{a}\|_p \leq R + S$, and so $\mathbf{a} \in \ell^p$. □

So, we have shown that the putative limit $\mathbf{a}$ (the point-wise limit of the sequence $(\mathbf{a}^k)_{k=1}^{\infty}$ of points in ℓ^p) is actually an element of the metric space ℓ^p. But we have yet to show that it is the *limit* of the sequence $(\mathbf{a}^k)$ *in* ℓ^p. That somewhat involved proof now follows.

Proposition 2. *Let $(\mathbf{a}^k)_{k=1}^{\infty}$ be a Cauchy sequence in ℓ^p, and let $\mathbf{a}$ be its point-wise limit (which is in ℓ^p, by Lemma 1). Then $\|\mathbf{a}^k - \mathbf{a}\|_p \to 0$ as $k \to \infty$.*

Proof. Let $\epsilon > 0$. Lemma 1 shows that $\mathbf{a} \in \ell^p$, which means that $\sum_{n=1}^{\infty} |a_n|^p < \infty$. Hence, by the Cauchy criterion, there is an $N_1 \in \mathbb{N}$ so that

$$\sum_{n=N_1}^{\infty} |a_n|^p < \epsilon^p.$$

In addition, we know that $(\mathbf{a}^k)_{k=1}^{\infty}$ is ℓ^p-Cauchy, so there is N_2 so that, whenever $k, m \geq N_2$, $\|\mathbf{a}^k - \mathbf{a}^m\|_p < \epsilon$. Letting $N = \max\{N_1, N_2\}$, we therefore have

$$\sum_{n=N}^{\infty} |a_n|^p < \epsilon^p \text{ and } \|\mathbf{a}^N - \mathbf{a}^k\|_p < \epsilon \;\; \forall k \geq N. \tag{3}$$

Now, the sequence $\mathbf{a}^N$ is in ℓ^p, and so we can apply the Cauchy criterion again: select N' large enough so that

$$\sum_{n=N'}^{\infty} |a_n^N|^p < \epsilon^p. \tag{4}$$

Note, we can always increase N' and still maintain this estimate, so we are free to chose $N' \geq N$.

We now use the constant N' we defined above in the bounds we will need later. Since $a_n^k \to a_n$ for each fixed n, we can choose K_1 so that $|a_1^k - a_1| < \epsilon^p/N'$ for $k \geq K_1$. Likewise, we can choose K_2 so that $|a_2^k - a_2| < \epsilon^p/N'$ for $k \geq K_2$. Continuing this way for N' steps, we can take $K = \max\{K_1, K_2, \ldots, K_{N'}\}$ and then we have

$$|a_n^k - a_n| < \frac{\epsilon^p}{N'}, \;\text{ for } k \geq K \text{ and } n \leq N'. \tag{5}$$

For good measure, we will also (increasing it if necessary) make sure that $K \geq N'$. Now, for any $k \geq K$, break up $\mathbf{b} = \mathbf{a}^k - \mathbf{a}$ as follows:

$$(b_n)_{n=1}^{\infty} = (b_n)_{n=1}^{N'-1} + (b_n)_{n=N'}^{\infty}.$$

(To be a little more pedantic, we are expressing $b_n = x_n + y_n$ where $x_n = b_n$ when $n < N'$ and $= 0$ when $n \geq N'$, and $y_n = 0$ when $n < N'$ and $= b_n$ when $n \geq N'$.) The triangle inequality for the p-norm then gives

$$\|\mathbf{a}^k - \mathbf{a}\|_p \leq \left(\sum_{n=1}^{N'-1} |a_n^k - a_n|^p\right)^{1/p} + \left(\sum_{n=N'}^{\infty} |a_n^k - a_n|^p\right)^{1/p}. \tag{6}$$

Equation 5 shows that, for $k \geq K$, the first term here is

$$\left(\sum_{n=1}^{N'-1} |a_n^k - a_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{N'-1} \frac{\epsilon^p}{N'}\right)^{1/p} = \left(\frac{N'-1}{N'}\right)^{1/p} \epsilon < \epsilon.$$

For the second term in Equation 6, we use the triangle inequality for the ℓ^p-norm restricted to the range $n \geq N'$ to get

$$\left(\sum_{n=N'}^{\infty} |a_n^k - a_n|^p\right)^{1/p} \leq \left(\sum_{n=N'}^{\infty} |a_n^k|^p\right)^{1/p} + \left(\sum_{n=N'}^{\infty} |a_n|^p\right)^{1/p}.$$

Since $N' \geq N$, Equation 3 shows that the second term here is $< \epsilon$. So, summing up the last two estimates, we have

$$\|\mathbf{a}^k - \mathbf{a}\|_p \leq 2\epsilon + \left(\sum_{n=N'}^{\infty} |a_n^k|^p\right)^{1/p}, \tag{7}$$

whenever $k \geq K$. So we need only show this final term is small. Here we make one more decomposition: $a_n^k = a_n^k - a_n^N + a_n^N$, and so once again applying the triangle inequality,

$$\left(\sum_{n=N'}^{\infty} |a_n^k|^p\right)^{1/p} \leq \left(\sum_{n=N'}^{\infty} |a_n^k - a_n^N|^p\right)^{1/p} + \left(\sum_{n=N'}^{\infty} |a_n^N|^p\right)^{1/p}.$$

The first of these terms is a sum of non-negative terms over $n \geq N'$, and so it is bounded above by the sum over $n \geq 1$ which is equal to $\|\mathbf{a}^k - \mathbf{a}^N\|_p$, which is $< \epsilon$ by Equation 3 (since $k \geq K \geq N' \geq N$). And the second term is also $< \epsilon$, by Equation 4. Whence, the last term in Equation 7 is also $< 2\epsilon$, and so we have shown that

$$\forall \epsilon > 0,\ \exists K \in \mathbb{N} \text{ such that } \forall k \geq K\ \|\mathbf{a}^k - \mathbf{a}\|_p < 4\epsilon.$$

Of course, we should have been more clever and chosen all our constants in terms of $\epsilon/4$ to get a clean ϵ in the end, but such tidying is not really necessary; 4ϵ is also arbitrarily small, and so we have shown that $(\mathbf{a}^k)_{k=1}^{\infty}$ does converge to $\mathbf{a}$ in ℓ^p. This concludes the proof that ℓ^p is complete. Whew! □

Let us conclude by remarking that a very similar (though somewhat simpler) proof works for $p = \infty$; the details are left to the reader.

18.100B Analysis I
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