COMPACTNESS VS. SEQUENTIAL COMPACTNESS

The aim of this handout is to provide a detailed proof of the equivalence between the two definitions of compactness: existence of a finite subcover of any open cover, and existence of a limit point of any infinite subset.

Definition 1. *K is* compact *if every open cover of K contains a finite subcover. K is* sequentially compact *if every infinite subset of K has a limit point in K.*

Theorem 1. *K is compact $\iff$ K is sequentially compact.*

The first half of this statement (compact $\Longrightarrow$ sequentially compact) is Theorem 2.37 in Rudin and is proved there. Our aim is to prove the converse implication (sequentially compact $\Longrightarrow$ compact), following the lines of Exercises 23, 24 and 26 in Rudin Chapter 2.

The proof requires the introduction of two auxiliary notions:

Definition 2. *A space X is* separable *if it admits a countable dense subset.*

For example $\mathbb{R}$ is separable ($\mathbb{Q}$ is countable, and it is dense since every real number is a limit of rationals); for the same reason $\mathbb{R}^k$ is separable (consider all points with only rational coordinates).

Definition 3. *A collection $\{V_\alpha\}$ of open subsets of X is said to be a* base *for X if the following is true: for every $x \in X$ and for every open set $G \subset X$ such that $x \in G$, there exists α such that $x \in V_\alpha \subset G$.*

In other words, every open subset of X decomposes as a union of a subcollection of the V_α's – the V_α's "generate" all open subsets. The family $\{V_\alpha\}$ almost always contains infinitely many members (the only exception is if X is finite). However, if X happens to be separable, then countably many open subsets are enough to form a base (the converse statement is also true and is an easy exercise):

Lemma 1. *Every separable metric space has a countable base.*

Proof. Assume X is separable: by definition it contains a countable dense subset $P = \{p_1, p_2, \dots\}$. Consider the countable collection of neighborhoods $\{N_r(p_i),\ r \in \mathbb{Q},\ i = 1, 2, \dots\}$. We show that it is a base by checking the definition.

Consider any open set $G \subset X$ and any point $x \in G$. Since G is open, there exists $r > 0$ such that $N_r(x) \subset G$. Decreasing r if necessary we can assume without loss of generality that r is rational. Since P is dense, by definition x is a limit point of P, so $N_{r/2}(x)$ contains a point of P. So there exists i such that $d(x, p_i) < \frac{r}{2}$. Since r is rational, the neighborhood $N_{r/2}(p_i)$ belongs to the chosen collection. Moreover, $N_{r/2}(p_i) \subset N_r(x) \subset G$. Finally, since $d(x, p_i) < \frac{r}{2}$ we also have $x \in N_{r/2}(p_i)$. So the chosen collection is a base for X. □

Lemma 2. *If X is sequentially compact then it is separable.*

Proof. Fix $\delta > 0$, and let $x_1 \in X$. Choose $x_2 \in X$ such that $d(x_1, x_2) \geq \delta$, if possible. Having chosen $x_1, \dots, x_j$, choose x_{j+1} (if possible) such that $d(x_i, x_{j+1}) \geq \delta$ for all $i = 1, \dots, j$. We first notice that this process has to stop after a finite number of iterations. Indeed, otherwise we would obtain an infinite sequence of points x_i mutually distant by at least δ; since X is sequentially compact the infinite subset $\{x_i,\ i = 1, 2, \dots\}$ would admit a limit point y, and the neighborhood $N_{\delta/2}(y)$ would contain infinitely many of the x_i's, contradicting the fact that any two of them are distant by at least δ. So after a finite number of iterations we obtain $x_1, \dots, x_j$ such that $N_\delta(x_1) \cup \dots N_\delta(x_j) = X$ (every point of X is at distance $< \delta$ from one of the x_i's).

We now consider this construction for $\delta = \frac{1}{n}$ ($n = 1, 2, \dots$). For $n = 1$ the construction gives points $x_{11}, \dots, x_{1j_1}$ such that $N_1(x_{11}) \cup \dots \cup N_1(x_{1j_1}) = X$, for $n = 2$ we get $x_{21}, \dots, x_{2j_2}$ such that

$N_{1/2}(x_{21} \cup \dots \cup N_{1/2}(x_{2j_2}) = X$, and so on. Let $S = \{x_{ki}, k \geq 1, 1 \leq i \leq j_k\}$: clearly S is countable. We claim that S is dense (i.e. $\bar{S} = X$). Indeed, if $x \in X$ and $r > 0$, the neighborhood $N_r(x)$ always contains at least a point of S (choosing n so that $\frac{1}{n} < r$, one of the x_{ni}'s is at distance less than r from x), so every point of X either belongs to S or is a limit point of S, i.e. $\bar{S} = X$. □

At this point we know that every sequentially compact set has a countable base. We now show that this is enough to get *countable* subcovers of any open cover.

Lemma 3. *If X has a countable base, then every open cover of X admits an at most countable subcover.*

Proof. Homework

□

The final ingredient is the following:

Lemma 4. *If $\{F_n\}$ is a sequence of non-empty closed subsets of a sequentially compact set K such that $F_n \supset F_{n+1}$ for all $n = 1, 2, \dots$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

(Since we know at this point that every compact set is sequentially compact, and since compact subsets are closed, this lemma implies immediately the Corollary to Theorem 2.36 in Rudin).

Proof. Take $x_n \in F_n$ for each integer n, and let $E = \{x_n,\ n = 1, 2, \dots\}$. If E is finite then one of the x_i belongs to infinitely many F_n's. Since $F_1 \supset F_2 \supset \dots$, this implies that x_i belongs to every F_n, and we get that $\bigcap_{n=1}^{\infty} F_n$ is not empty.

Assume now that E is infinite. Since K is sequentially compact, E has a limit point y. Fix a value of n: every neighborhood of y contains infinitely many points of E; among them, we can find one which is of the form x_i for $i \geq n$ and therefore belongs to F_n (because $x_i \in F_i \subset F_n$). Since every neighborhood of y contains a point of F_n, we get that either $y \in F_n$, or y is a limit point of F_n; however since F_n is closed, every limit point of F_n belongs to F_n. So in either case we conclude that $y \in F_n$. Since this holds for every n, we obtain that $y \in \bigcap_{n=1}^{\infty} F_n$, which proves that the intersection is not empty. □

We can now prove the theorem. Assume that K is sequentially compact, and let $\{G_\alpha\}$ be an open cover of K. By Lemma 1 and Lemma 2, K has a countable base, so by Lemma 3 $\{G_\alpha\}$ admits an at most countable subcover that we will denote $\{G_i\}_{i \geq 1}$. Our aim is to show that $\{G_i\}$ admits a finite subcover (which will also be a finite subcover of $\{G_\alpha\}$). If $\{G_i\}$ only contains finitely many members, we are already done; so assume that there are infinitely many G_i's, and assume that for every value of n we have $G_1 \cup \dots \cup G_n \not\supset K$ (else we have found a finite subcover).

Let $F_n = \{x \in K,\ x \notin G_1 \cup \dots \cup G_n\} = K \cap G_1^c \cap \dots \cap G_n^c$. Because the G_i are open, F_n is closed; by assumption F_n is non-empty; and clearly $F_n \supset F_{n+1}$ for all n. Therefore, applying Lemma 4 we obtain that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, i.e. there exists a point $y \in K$ such that $y \notin G_1 \cup \dots \cup G_n$ for every n. We conclude that $y \notin \bigcup_{i=1}^{\infty} G_i$, which is a contradiction since the open sets G_i cover K.

So there exists a value of n such that $G_1, \dots, G_n$ cover K. We conclude that every open cover of K admits a finite subcover, and therefore that K is compact.

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