Problem Set 1 Solutions, 18.100C, Fall 2012

September 12, 2012

1

Let m and n be positive integers with no common factor. Prove that if $\sqrt{m/n}$ is rational, then m and n are both perfect squares, that is to say there exist integers p and q such that $m = p^2$ and $n = q^2$. (This is proved in Proposition 9 of Book X of Euclids Elements).

Assume $\sqrt{m/n}$ is rational. Then there exist positive integers M and N with no common factor such that $\sqrt{m/n} = M/N$ and so $mN^2 = nM^2$.

Claim: M^2 divides m and N^2 divides n.

Assume the claim for now. Then

$$m = M^2 m' \text{ and } n = N^2 n' \text{ for some } m' \text{ and } n'.$$

Substituting we obtain $M^2 m' N^2 = N^2 n' M^2$ which gives $m' = n'$. $m' = n'$ divides m and n so $m' = n' = 1$ and we have shown m and n are perfect squares.

Proof of claim: We show that M^2 divides m; the argument that N^2 divides n is identical. Write M as a product of primes $p_1 \cdots p_r$ and note that no p_i divides N. Assume inductively that $p_1^2 \cdots p_t^2$ divides m. Then

$$p_{t+1}^2 \;\Big|\; \frac{M^2}{p_1^2 \cdots p_t^2} \;\Big|\; \frac{m}{p_1^2 \cdots p_t^2} N^2$$

Since p_{t+1} does not divide N^2 we see

$$p_{t+1}^2 \;\Big|\; \frac{m}{p_1^2 \cdots p_t^2}, \text{ which gives } p_1^2 \cdots p_{t+1}^2 \;\Big|\; m.$$

The inductive hypothesis holds when $t = 0$; the empty product is 1. Thus, by induction $p_1^2 \cdots p_r^2 = M^2$ divides m.

2

Problem 8 from Page 22: Prove that no order can be defined in the complex field that turns it into an ordered field.

Suppose that such an order exists. We know that in any ordered field squares are greater than or equal to zero. Since $i^2 = -1$, this means that $0 \leq -1$. Thus

$$1 = 0 + 1 \leq -1 + 1 = 0 \leq 1,$$

which implies $0 = 1$, a contradiction.

3

Problem 9 from Page 22: suppose $z = a + bi$ and $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. Does this set have the least-upper-bound property?

Suppose $z = a + bi$ and $w = c + di$ are distinct complex numbers. If $a \neq c$, then either $a > c$ or $a < c$, and hence either $z < w$ or $z > w$. If $a = c$, then $b \neq d$, since otherwise $z = w$. In this case, either $b < d$ or $b > d$, and then $z > w$ or $z < w$. So in either case either $z < w$ or $z > w$.

Now suppose $x = a + bi$, $y = c + di$, and $z = e + fi$ are complex numbers with $x < y$ and $y < z$. We need to show that $x < z$. We have $a \leq c$, since otherwise $x > y$, and similarly $c \leq e$, and so $a \leq e$. If $a < e$, then $x < z$. If $a = e$, then in fact $a = c = e$. In this case, since $x < y$, we must then have $b < d$ and similarly $d < f$, so $b < f$ and again $x < z$. In either case $x < z$.

So this relation makes $\mathbf{C}$ into an ordered set. This order does not have the least-upper-bound property. To see this, consider the purely imaginary line, i.e. the set

$$L = \{a + bi \in \mathbf{C} | a = 0\}$$

This set is bounded above by 1. Let $z = a + bi$ be any upper bound for L. Then $a > 0$; indeed, if $a \leq 0$, then the complex number $w = (b+1)i$ satisfies $z < w$ and $w \in L$, so z cannot be an upper bound for L. Now consider the complex number $y = a + (b - 1)i$. Clearly $y < z$. However, since $a > 0$ y

is also an upper bound for L. This shows that for every upper bound on L there exists a strictly smaller upper bound, and so $\mathbf{C}$ cannot have the least-upper-bound property with this ordering.

4

Let $\mathbf{R}$ be the set of real numbers and suppose $f : \mathbf{R} \to \mathbf{R}$ is a function such that for all real numbers x and y the following two equations hold
$(1) f(x + y) = f(x) + f(y)$
$(2) f(xy) = f(x)f(y)$

Claim: $f(x) = 0$ for all x or $f(x) = x$ for all x.

a

Setting $x = 1$ and $y = 0$ in equation (1) gives

$$f(1) = f(1) + f(0) \text{ so that } f(0) = 0.$$

Setting $x = y = 1$ in equation (2) gives $f(1) = f(1)^2$. Thus $f(1)$ is equal to 0 or 1 as we may see by solving the equation $x - x^2 = x(1 - x) = 0$.

Remark: If I did not have to answer all parts of the question fully and I just wanted to prove the claim as quickly as possible I would proceed by noting that $f(1) = 0$ gives

$$f(x) = f(1)f(x) = 0 \text{ for all } x \in \mathbf{R}.$$

From this moment on I could then assume $f(1) = 1$.

b

By a) we have

$$f(0) = 0 = 0f(1).$$

Also,

$$f(x) + f(-x) = f(x - x) = f(0) = 0 \text{ so that } f(-x) = -f(x) \text{ for all } x \in \mathbf{R}.$$

In particular, $f(-1) = -f(1)$.

Let $n \in \mathbf{Z}$ and assume that $f(n) = nf(1)$. Then

$$f(n + 1) = f(n) + f(1) = nf(1) + f(1) = (n + 1)f(1)$$

and

$$f(n-1) = f(n) + f(-1) = nf(1) - f(1) = (n-1)f(1).$$

By induction $f(n) = nf(1)$ for all $n \in \mathbf{Z}$.

For $n, m \in \mathbf{Z}$, $m \neq 0$

$$f\left(\frac{n}{m}\right) mf(1) = f\left(\frac{n}{m}\right) f(m) = f(n) = nf(1)$$

and so

$$f\left(\frac{n}{m}\right) = f\left(\frac{n}{m}\right) f(1) = \frac{n}{m} f(1).$$

Thus $f(q) = qf(1)$ for all $q \in \mathbf{Q}$ and by $a)$ either $f(q) = 0$ for all $q \in \mathbf{Q}$ or $f(q) = q$ for all $q \in \mathbf{Q}$.

c

Suppose $x \geq 0$. Then there exists a $y \in \mathbf{R}$ such that $y^2 = x$ and

$$f(x) = f(y^2) = f(y)^2 \geq 0.$$

Thus

$$x \geq y \Rightarrow x - y \geq 0 \Rightarrow f(x) - f(y) = f(x) + f(-y) = f(x-y) \geq 0 \Rightarrow f(x) \geq f(y).$$

d

Suppose $f(1) = 0$. Given any $x \in \mathbf{R}$ we can find $p, q \in \mathbf{Q}$ such that

$$p \leq x \leq q.$$

Then

$$0 = f(p) \leq f(x) \leq f(q) = 0 \Rightarrow f(x) = 0.$$

Alternatively, we proceed as remarked in $a)$.

Suppose $f(1) = 1$. Let $x \in \mathbf{R}$ and $n \in \mathbf{N}$. Then there exist $p, q \in \mathbf{Q}$ such that

$$x - \frac{1}{n} \leq p \leq x \leq q \leq x + \frac{1}{n}$$

and so

$$x - \frac{1}{n} \leq p = f(p) \leq f(x) \leq f(q) = q \leq x + \frac{1}{n}.$$

So for all $x \in \mathbf{R}$ and all $n \in \mathbf{N}$ we have

$$x - \frac{1}{n} \leq f(x) \leq x + \frac{1}{n}$$

which gives $f(x) = x$ for all $x \in \mathbf{R}$.

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