Problem Set 10 Solutions, 18.100C, Fall 2012

December 5, 2012

1

We have a continuous $K : [0,1] \times [0,1] \to \mathbb{R}$ which is continuous. For each fixed $x \in [0,1]$, the function $y \mapsto K(x,y)$ is thus a continuous function from $[0,1] \to \mathbb{R}$, hence is Riemann-integrable. Since $f : [0,1] \to \mathbb{R}$ is Riemann-integrable by assumption and the product of two integrable functions is integrable (Rudin 6.13), we conclude that

$$g(x) = \int_0^1 K(x,y)f(y)dy$$

Is well-defined. We will show that it is continuous.

Let $\epsilon > 0$. By Rudin 6.13, $|f|$ is also Riemann-integrable. Pick $M \in \mathbb{R}$ sufficiently large that

$$\int_0^1 |f(y)|dy < M$$

K is a continuous function on the compact set $[0,1] \times [0,1]$, hence is uniformally continuous. Thus, there exists $\delta > 0$ such that

$$\sqrt{(x'-x)^2 + (y'-y)^2} < \delta \implies |K(x',y') - K(x,y)| < \frac{\epsilon}{M}$$

For $\{x, x', y, y'\} \subset [0,1]$. In particular,

$$|x'-x| < \delta \implies |K(x',y) - K(x,y)| < \frac{\epsilon}{M}$$

Thus, for $|x'-x| < \delta$, we have

$$|g(x') - g(x)| = \left| \int_0^1 K(x', y)f(y)dy - \int_0^1 K(x, y)f(y)dy \right|$$
$$= \left| \int_0^1 (K(x', y) - K(x, y))f(y)dy \right| \leq \int |K(x', y) - K(x, y)| \cdot |f(y)|dy$$
$$< \frac{\epsilon}{M} \int_0^1 |f(y)|dy < \epsilon$$

Which proves the (uniform) continuity of g.

2

Let $a, b \in \mathbb{R}$. I claim that $\max(a, b) = (a+b+|a-b|)/2$. There are three cases.

$a = b$: Then $(a + b + |a - b|)/2 = a = \max(a, b)$.

$a > b$: Then $(a + b + |a - b|)/2 = (a + b + a - b)/2 = a = \max(a, b)$.

$a < b$: Then $(a + b + |a - b|)/2 = (a + b + b - a)/2 = b = \max(a, b)$.

Now suppose f and g are Riemann-Stieltjes integrable for some α. Then $f-g$ is integrable, hence so is $|f-g|$, hence so is $(f+g+|f-g|)/2 = \max(f, g)$.

3

a

Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then by theorem 6.8, $f^2 \in \mathcal{R}$. By assumption there exists $x_0 \in [a, b]$ with $f(x_0) \neq 0$. Because f^2 is continuous at x_0 there exists a $\delta > 0$ such that

$$x \in [a, b],\ |x - x_0| < \delta \implies |f^2(x) - f^2(x_0)| < |f^2(x_0)|/2.$$

Thus

$$x \in [a, b],\ |x - x_0| < \delta \implies |f^2(x)| > |f^2(x_0)|/2.$$

By definition

$$\int_a^b f^2(x)dx = \sup L(P, f^2) \quad \text{and} \quad L(P, f^2) = \sum_{i=1}^{n} m_i \Delta x_i.$$

If we choose a partition P with two points in $[a, b] \cap (x_0 - \delta, x_0 + \delta)$ then at least one of the m_i's is strictly greater than zero and the rest are all greater than or equal to zero. Thus for this partition $L(P, f) > 0$ and we obtain $\int f^2 > 0$.

b

Let $f : [0, 1] \to \mathbb{R}$ be continuous and suppose that $\int_0^1 x^n f(x)dx = 0$ for all $n \in \mathbb{N}$. From linearity of the integral we deduce that $\int_0^1 p(x)f(x)dx = 0$ for all polynomials p. Weierstrass' theorem says that we can find a sequence of polynomials (p_n) converging uniformly to f. Since f is bounded $(p_n f)$ converges uniformly to f^2. Then

$$\int_0^1 p_n(x)f(x)dx \to \int_0^1 f(x)^2 dx$$

giving $\int_0^1 f(x)^2 dx = 0$. Since f^2 is positive and continuous this implies that $f^2 = 0$ and so $f = 0$.