Problem Set 2 Solutions, 18.100C, Fall 2012

September 20, 2012

1

We define

$$F = \mathbb{Q} = \{a + \sqrt{2}b | a, b \in \mathbb{Q}\} \subset \mathbb{R}$$

We wish to show that F is a subfield of $\mathbb{R}$. In order to show this, we need to show that a) $0, 1 \in F$; b) F is closed under addition and multiplication; and c) if $x \in F$ and $x \neq 0$, then $-x \in F$ and $1/x \in F$. The commutative, associate, and distributive properties all follow from the corresponding properties on $\mathbb{R}$.

a), b), and the first half of c) are straightforward; we have $0 = 0 + 0\sqrt{2} \in F$ and $1 = 1 + 0\sqrt{2} \in F$. For b), we have

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in F$$

and

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ab + 2cd) + (ad + bc)\sqrt{2} \in F.$$

If $x = a + b\sqrt{2}$, then $-x = (-a) + (-b)\sqrt{2} \in F$. So the only fact remaining to show is that F is closed under multiplicative inverses.

To prove this, we need the following

Fact: if $0 = a + b\sqrt{2} \in F$, then $a = b = 0$

Proof: Suppose $b \neq 0$. Then $\sqrt{2} = -a/b \in \mathbb{Q}$, a contradiction. So we must have $b = 0$, and then $0 = a + 0 = a$.

Now take $x = a + b\sqrt{2} \in F$, $x \neq 0$. By the above fact, $a - b\sqrt{2}$ is also nonzero, and hence

$$a^2 - 2b^2 = (a + b\sqrt{2})(a - b\sqrt{2}) \neq 0$$

Since the product of non-zero real numbers is non-zero.

So we can define $c = a/(a^2 - 2b^2) \in \mathbb{Q}$, $d = -b/(a^2 - 2b^2) \in \mathbb{Q}$, and $y = c + d\sqrt{2} \in F$. I claim that $xy = 1$, so $y = 1/x$ and F contains multiplicative inverses. Indeed,

$$(a + b\sqrt{2})(c + d\sqrt{2}) = \frac{1}{a^2 - 2b^2}(a + b\sqrt{2})(a - b\sqrt{2}) = \frac{a^2 - 2b^2}{a^2 - 2b^2} = 1$$

and we are done.

2

Problem 11 from page 23.

Let $z = a + bi \in \mathbb{C}$. We wish to show that $z = rw$, where $r \geq 0$ is a positive real number and w is a complex number with $|w| = 1$. Suppose $z = 0$. Then we can take $r = 0$ and $w = 1$. If $z \neq 0$, we take $r = |z| > 0$, ($r > 0$ by theorem 1.33(a)) and take $w = z/r$. Then obviously $z = rw$, and

$$|w| = |\frac{z}{r}| = \frac{|z|}{|z|} = 1$$

by theorem 1.33(c). As for uniqueness, r is always determined by z; indeed, if $z = rw$, we must have $|z| = |rw| = |r| \cdot |w| = |r| = r$, since $r \geq 0$. If $z = 0$, w is not determined by z, since for any w, $rw = 0w = 0 = z$. However, if $z \neq 0$, then $r \neq 0$, and then we must have $w = z/r$. So w is determined by z so long as $z \neq 0$.

3

Problem 9 from page 43.
Let X be a metric space and $E \subset X$.

a

Let $p \in E^\circ$. By definition, p is an interior point of E, so there exists an $r > 0$ such that $N_r(p) \subset E$. If we can show $N_r(p) \subset E^\circ$ it will follow that p is an interior point of E°, and thus E° is open. But for any $q \in N_r(p)$ we have

$$N_{r-d(p,q)}(q) \subset N_r(p) \subset E,$$

which implies $q \in E^\circ$, as required.

(For the above inclusion we use the triangle inequality:

$$x \in N_{r-d(p,q)}(q) \implies d(x,q) < r - d(p,q) \implies d(x,p) \leq d(x,q) + d(p,q) < r$$

$$\implies x \in N_r(p).)$$

b

E is open $\iff$ every point of E is an interior point of $E \iff E \subset E^\circ$.

It is clear that we always have $E^\circ \subset E$ (since a neighborhood of a point contains the point). Hence, E is open if and only if $E^\circ = E$.

c

Let $G \subset E$ and suppose G is open. Given $p \in G$, there exists an $r > 0$ such that $N_r(p) \subset G$. Since $G \subset E$ we have

$$N_r(p) \subset E$$

and so $p \in E^\circ$.

d

By definition, $x \in E^\circ$ if and only if there exists an $r > 0$ such that $N_r(x) \subset E$. Thus, $x \notin E^\circ$ if and only if for all $r > 0$, $N_r(x) \cap (X \setminus E) \neq \emptyset$.

Suppose that for all $r > 0$, $N_r(x) \cap (X \setminus E) \neq \emptyset$. Then either $x \in X \setminus E$

or x is a limit point of $X \setminus E$, i.e. $x \in \overline{X \setminus E}$. Conversely, if $x \in \overline{X \setminus E}$, then either $x \in X \setminus E$ or x is a limit point of $X \setminus E$ and in either case $N_r(x) \cap (X \setminus E) \neq \emptyset$, for all $r > 0$.

e, f

No, in both cases. Let $X = \mathbb{R}$ and $E = \mathbb{Q}$.

Claim: $E^\circ = \emptyset$ and $\overline{E} = X$.

Proof: Let $x \in X$. Then for each $r > 0$, there exists a $q_r \in E$ with $x < q_r < x + r$. Thus

$$q_r \in (N_r(x) \setminus \{x\}) \cap E \neq \emptyset$$

for each $r > 0$. This says x is a limit point of E and so $x \in \overline{E}$, giving $\overline{E} = X$. Similarly, $\overline{X \setminus E} = X$ and so $X \setminus E^\circ = X$, which gives $E^\circ = \emptyset$.

One easily sees $\overline{E^\circ} = \emptyset$ and $(\overline{E})^\circ = X$ and so we have counterexamples.

4

Problem 29 from page 45

Let $A \subset \mathbb{R}$ open. We will first show that A can be written as a union of disjoint open intervals, and then show that this collection of intervals is necessarily countable.

Let $x \in A$. We define the sets L_x and U_x by

$$L_x = \{y \in \mathbb{R} | y \leq x, [y, x] \subset A\}, U_x = \{y \in \mathbb{R} | y \geq x, [x, y] \subset A\}$$

Since A is open, for all $\epsilon > 0$ sufficiently small, $x \pm \epsilon \in A$, so L_x and U_x both contain elements other than x. Note that $L_x, U_x \subset A$. Let $c_x = \inf L_x$, and $d_x = \sup U_x$; it is possible that c_x could be $-\infty$, and d_x could be ∞.

Claim 1: $(c_x, x] \subset A$ and $[x, d_x) \subset A$

Proof: If $y \in (c_x, x)$, then since c_x is the inf of L_x, y cannot a lower bound for L_x. So there is some y' with $c_x < y' < y$ such that $y' \in L_x$, or $[y', x] \subset A$, which implies $y \in A$ (in fact $y \in L_x$). The same argument applies for d_x.

Claim 2: $c_x \notin A$ and $d_x \notin A$

Proof: This is immediate if $c_x = -\infty$. So suppose $c_x > -\infty$ and $c_x \in A$. Since A is open, there exists an $\epsilon > 0$ such that $[c_x - \epsilon, c] \subset A$ (take any $\epsilon < r$ where $B_r(c_x) \subset A$.) But then $[c_x - \epsilon, c_x] \cup [c_x, x] = [c_x - \epsilon, x] \subset A$, so $c_x - \epsilon \in L_x$, which is a contradiction since c_x is the inf of L_x. The same argument applies for d_x.

Claim 3: $(c_x, x] = L_x$ and $[x, d_x) = U_x$

Proof: The proof of claim 1 shows that $(c_x, x) \subset L_x$. Conversely, if $y \in L_x$, then $c_x \leq y \leq x$, and by claim 2 $y \neq c_x$, so $c_x < y$ and $y \in (c_x, x]$.

We can now define $E_x = L_x \cup U_x = (c_x, d_x)$; one should think of E_x as the largest open interval around x contained in A. Note that $E_x \subset A$, so $\bigcup_{x \in A} E_x \subset A$, and conversely if $x \in A$ then $x \in E_x \subset \bigcup_{x \in A} E_x$, and so $A = \bigcup_{x \in A} E_x$.

Claim 4: If $x, y \in A$, then either $E_x = E_y$ or $E_x \cap E_y = \emptyset$.

Proof: Suppose $E_x \neq E_y$, and write $E_x = (c, d)$ and $E_y = (e, f)$. Without loss of generality assume $c \leq e$. If $e = c$, then $d \neq f$; without loss of generality $d < f$. Then $d \in E_y \subset A$; however, by claim 2 $d \notin A$, a contradiction. So we can assume $c < e$. If $e < d$, then $e \in E_x \subset A$; however, again by Claim 2 $e \notin A$, a contradiction. So $e \geq d$, which implies that (e, f) is disjoint from (c, d).

In other words, let $\mathcal{U} = \{E_x | x \in A\}$. Then $\mathcal{U}$ is a collection of open intervals whose union is equal to A; by Claim 4 all of the intervals in $\mathcal{U}$ are disjoint (think carefully about what Claim 4 says if this isn't obvious to you.)

We still have to show that $\mathcal{U}$ is countable (by countable I mean either finite or countably infinite.) We will do so by defining an injective map $f : \mathcal{U} \to \mathbb{Q}$. Let $E \in \mathcal{U}$. Then E is an open interval (c, d); pick a rational number $q_E \in (c, d) = E$. Make such a choice for every interval in $\mathcal{U}$.(*) We define the map $f : \mathcal{U} \to \mathbb{Q}$ by $f(E) = q_E$.

Claim 5: f is injective

Proof: Suppose $f(E) = f(E')$. Then $q_E \in E \cap E'$ by the definition of f. But the intervals in $\mathcal{U}$ are disjoint, so $E = E'$.

Thus, via f, $\mathcal{U}$ is bijective to a subset of $\mathbb{Q}$. But $\mathbb{Q}$ is countable, and by theorem 2.8 in Rudin every subset of a countable set is countable. Hence $\mathcal{U}$ is countable.

(*) For those who know some Set Theory, you need to the full Axiom of Choice to make these choices. If you don't know what that means, don't worry.