Problem Set 3 Solutions, 18.100C, Fall 2012

September 26, 2012

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We have a metric space (X, d), and define the function $d'(x,y) = \sqrt{d(x,y)}$. We wish to show that (X, d') is also a metric space with the same open sets as (X, d). We first check that d' is a metric.

(a) If $x \neq y$, then $d'(x,y) = \sqrt{d(x,y)} > 0$ since $d(x,y) > 0$, and similarly $d'(x,x) = 0$.

(b) $d'(x,y) = \sqrt{d(x,y)} = \sqrt{d(y,x)} = d'(y,x)$

(c) For the triangle inequality, we first need the following elementary

Fact: If $a, b \geq 0$, then $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

Indeed, squaring the right hand side gives $a + b + 2\sqrt{ab} \geq a + b$, and the square root function is order preserving. Using this fact, for $x, y, z \in X$ we have

$$d'(x,z) = \sqrt{d(x,z)} \leq \sqrt{d(x,y) + d(y,z)} \leq \sqrt{d(x,y)} + \sqrt{d(y,z)} = d'(x,y) + d'(y,z)$$

Now, let E be an open set for E. We need to show that it is open for d'. Let $x \in E$. Then there is some $r > 0$ such that the ball of radius r around x is contained in E, where the ball is taken with respect to d, i.e. $N_r(x) \subset E$. But the ball of radius r with respect to d is the ball of radius $\sqrt{r}$ with respect to d', so there is a neighbourhood of x with respect to d' contained in E. In other words, E is open with respect to d'. Similarly, a set that is open with respect to d' is also open with respect to d.

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We prove the result for $\mathbb{R}^n$.

Lemma: $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Proof: Just use the density of $\mathbb{Q}$ in $\mathbb{R}$ for each coordinate.

Theorem: Let $n \in \mathbb{N}$ and let $S \in \mathbb{R}^n$ be a set such that every point in S is isolated. Then S is at most countable.

Proof: Fix $s \in S$. Since s is an isolated point, there exists an $\tilde{r}_s > 0$ such that $N_{\tilde{r}_s}(s) \cap S = \{s\}$; let $r_s = \tilde{r}_s/2$ and pick an element $t_s \in N_{r_s}(s) \cap \mathbb{Q}^n$. Doing this for each s defines a function

$$f : S \to \mathbb{Q}^n, \ \ s \mapsto t_s.$$

We now go about showing that f is injective; since $\mathbb{Q}^n$ is countable this will show S is at most countable.

Suppose $f(s) = f(\tilde{s})$ and let $t = f(s)$. Then $t = t_s = t_{\tilde{s}} \in N_{r_s}(s) \cap N_{r_{\tilde{s}}}(\tilde{s})$. Thus

$$d(s, \tilde{s}) \leq d(t, s) + d(t, \tilde{s}) < r_s + r_{\tilde{s}} \leq \max\{\tilde{r}_s, \tilde{r}_{\tilde{s}}\}$$

so either $s \in N_{\tilde{r}_{\tilde{s}}}(\tilde{s})$ or $\tilde{s} \in N_{\tilde{r}_s}(s)$. In either case we obtain $s = \tilde{s}$.

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X is a space where every infinite subset has a limit point. We first prove the following

Lemma 1: Let $\delta > 0$. Then there exists a finite set N_δ with the following properties: (a) For every $x, y \in N_\delta$, $x = y$, $d(x, y) \geq 0$. (b) For every $z \in X$, there exists a $y \in N_\delta$ such that $d(z, y) < \delta$.

Fix $\delta > 0$. We construct N_δ inductively. Pick an arbitrary $x_1 \in X$. Assume we have $x_1, x_2 \ldots x_m$ with $d(x_i, x_j) \geq \delta$ for $i = j$. If every point of X is within δ of $\{x_1, \ldots x_m\}$, then we can take $N_\delta = \{x_1, \ldots x_m\}$ which satisfies (a) and (b) of the lemma. If not, we choose x_{m+1} such that $d(x_{m+1}, x_i) \geq \delta$ for $1 \leq i \leq m$.

We claim that this process must terminate at some finite M, at which point we are done. If not, then by this process we have constructed an infinite set $C = \{x_1, x_2, x_3 \ldots\}$ with $d(x_i, x_j) \geq \delta$ for $i \neq j$. By our assumption on X, this set has a limit point x. Now consider the open neighbourhood $N_{\delta/4}(x)$. This must contain two distinct points $x_i \neq x_j \neq x$ (in fact, infinitely many points, by Theorem 2.20.) Using the triangle inequality, we have

$$\delta \leq d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < \delta/4 + \delta/4 = \delta/2$$

A contradiction.

Using the above Lemma, for each $m \in \mathbb{N}$, we have a finite set $N_{1/m}$ such that every point of X is within $1/m$ of some point of $N_{1/m}$. Let $D = \cup_m N_{1/m}$. D is a countable union of finite sets, and hence is countable. We claim that D is dense. Take any $x \in X$, and $r > 0$. Pick an m sufficiently large that $1/m < r$. Then by definition there is a $y \in N_{1/m} \subset D$ such that $d(x, y) < 1/m$. But then $y \in N_r(x)$. Since x and r are arbitrary, this proves that D is dense.

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Let $A = \{p \in \mathbb{R} | p = d(x, f(x)$ for some $x \in X\}$. Since distances are non-negative A is bounded below by 0. Let $a = \inf A$. Obviously $a \geq 0$. We make the following claim, which we will prove later

Claim: There exists $x \in X$ such that $d(x, f(x)) = a$. In other words, the infimum is actually attained in A.

Now, assuming the claim, if $a = 0$, then we are done, since $0 = d(x, f(x))$ so $x = f(x)$ is a fixed point. So suppose $a > 0$. Then $x \neq f(x)$. Set $y = f(x)$. Then we have $d(y, f(y)) \in A$, and

$$d(y, f(y)) = d(f(x), f(y)) < d(x, y) = d(x, f(x)) = a$$

Which is a contradiction, since a is a lower bound for A. So $a = 0$ and we are done.

Proof of the claim: suppose the claim is false. Define the set $U_n = \{x \in$

$X | d(x, f(x)) > a + 1/n\}$. We claim that the sets U_n cover X. For any $x \in X$, since the inf is not attained, we must have $d(x, f(x)) = a + r$ where $r > 0$. Take $n \in \mathbb{N}$ sufficiently large that $r > 1/n$. Then $d(x, f(x)) > a + 1/n$, so $x \in U_n$ and the U_n's cover X.

We claim that U_n is open. To see this, let $x \in U_n$. Then $d(x, f(x) > a + 1/n$. Choose a small $\epsilon > 0$ such that $\epsilon < (d(x, f(x)) - a - 1/n)/2$; then $d(x, f(x)) - 2\epsilon > a + 1/n$. Then we have $N_\epsilon(x) \subset U_n$. To see this, let $y \in N_\epsilon(x)$. Note that $d(f(x), f(y)) < d(x, y) < \epsilon$ since f is contracting. Then using the triangle inequality twice, we have

$$d(x, f(x)) \leq d(x, y) + d(y, f(x)) \leq d(x, y) + d(y, f(y)) + d(f(y), f(x)) < \epsilon + d(y, f(y)) + \epsilon$$

Rearranging this, we get

$$d(y, f(y)) > d(x, f(x)) - 2\epsilon > a + 1/n$$

So $y \in U_n$. Thus we have showed that every point of U_n has a neighbourhood contained entirely in U_n, so U_n is open.

In other words, we have constructed an open cover $\{U_n\}$ of X. Since X is compact, this cover has a finite subcover $\{U_{n_1}, \ldots U_{n_m}\}$; assume we have labelled these such that $n_i < n_j$ for $i < j$. Note that the U_n are increasing, i.e. $U_m \subset U_n$ if $m < n$. Thus $U_{n_i} \subset U_{n_m}$, and so $\{U_{n_m}\}$ also covers X, i.e. $X = U_{n_m}$. But then for all $x \in X$, we have $d(x, f(x)) > a + 1/n_m$ by the definition of U_{n_m}. Thus $a + 1/n_m$ is a lower bound for A strictly larger than a, which contradicts the fact that $a = \inf A$. This proves the claim, and hence the result.

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