Problem Set 6 Solutions, 18.100C, Fall 2012

October 25, 2012

1

Let $s_k = \prod_{k=1}^n x_k$. Then we say that $\prod_{k=1}^\infty x_k$ converges to s if the sequence s_n converges to s as $n \to \infty$.

Now suppose $\prod_{k=1}^\infty x_k = s$ converges to some $s \neq 0$. We wish to show that $\lim_k x_k = 1$. Let $y_k = x_k - 1$; then this is equivalent to $\lim_k y_k = 0$. Note that

$$s_{n+1} - s_n = \prod_{k=1}^{n+1} x_k - \prod_{k=1}^{n} x_k = (x_{n+1} - 1)\prod_{k=1}^{n} x_k = y_{k+1}s_n$$

Now pick $\epsilon > 0$; we will find $N \in \mathbb{N}$ such that $n > N \implies |y_n| < \epsilon$, which will prove that $\lim_k y_k = 0$. Since $\lim_n s_n = s$ and $s \neq 0$, there exists an M such that $n > M \implies |s - s_n| < |s|/2$, which implies $|s_n| > |s|/2$. Let $\delta > 0$ be sufficiently small that $\epsilon|s|/2 > \delta$. Since s_n converges it is a Cauchy sequence, so there exists $N > M$ such that for $n, m > N$, $|s_n - s_m| < \delta$. In particular, for any $n > N$ we have

$$\delta > |s_{n+1} - s_n| = |y_{n+1}s_n| > |y_{n+1}| \cdot \frac{|s|}{2}$$

So $|y_{n+1}| < 2\delta/|s| < \epsilon$. So $N + 1$ works for this ϵ.

As for $\prod_{k=1}^\infty (1 + 1/k)$, we have

$$s_n = \prod_{k=1}^{n} (1 + \frac{1}{k} = 1 + \sum_{k=1}^{n} \frac{1}{k} + \sum_{1 \leq k_1, k_2 \leq N} \frac{1}{k_1 k_2} + \cdots \frac{1}{1 \cdot 2 \cdots (k-1)k}$$

$$> \sum_{k=1}^{n} \frac{1}{k}$$

Since the partial sums $\sum_{k=1}^{n} 1/k$ diverge to infinity, we must have $\lim_n s_n = \infty$, and so this product does not converge.

2

Here we adapt Rudin's proof of Theorem 3.27. Let $a_1 > a_2 > a_3 \cdots > 0$ be a decreasing sequence of positive real numbers. Let $b_n = \sum_{k=2^{n-1}}^{2^n-1} a_k$. We then have

$$\sum_{k=1}^{n} b_n = \sum_{k=1}^{2^n-1} a_k$$

So $\sum_k b_k$ converges if and only if $\sum_k a_k$ does, and converges to the same value. Since a_k is decreasing, we have

$$b_n = \sum_{k=2^{n-1}}^{2^n-1} a_k < \sum_{k=2^{n-1}}^{2^n-1} a_{2^{n-1}} = 2^{n-1} a_{2^{n-1}}$$

Now specialize to the case $a_k = 1/k^2$. Then $2^n a_{2^n} = 2^{n-2n} = 2^{-n}$. Thus we have the estimate

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} b_k < \sum_{k=0}^{\infty} 2^{-k} = 2$$

(Note the index shift), which is not quite as tight as we want. However, we can use the same idea to get sharper estimates. Indeed, note that

$$\sum_{k=5}^{\infty} b_k < \sum_{k=4}^{\infty} 2^{-k} = 2^{-3}$$

On the other hand, we can explicity compute

$$b_1 + b_2 + b_3 + b_4 = \sum_{k=1}^{15} \frac{1}{k^2} \approx 1.58 < 1.6$$

Of course one should give the precise fractional value, rather the than approximate decimal one, but I don't have Mathematica handy at the moment, and this estimate is sufficient.

Putting these two estimates together, we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{4} b_k + \sum_{k=5}^{\infty} b_k < 1.6 + 2^{-3} = 1.725 < 1.75 = 7/4$$

For those who are curious, the actual value is $\pi^2/6$, first calculated by Euler with an argument that is at the same time brilliant and sufficiently unrigorous that you would probably receive no credit if you wrote it up for this course.

3

We have a continuous function $f : X \to Y$, and $E \subset X$. We wish to show that $f(\overline{E}) \subset \overline{f(E)}$ Let $x \in \overline{E}$. Then $f(x) \in \overline{f(E)}$ if any only if, for every $\epsilon > 0$, $N_\epsilon(f(x)) \cap f(E) \neq \emptyset$.

So let $\epsilon > 0$. Since f is continuous at x, there exists $\delta > 0$ such that for all $y \in X$, $d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$. But $x \in \overline{E}$, so all neighbourhoods of X intersect E. In other words there exists $y \in E$ such that $d(x, y) < \delta$. Then $d(f(x), f(y)) < \epsilon$, so $f(y) \in N_\epsilon(f(x)) \cap f(E) \neq \emptyset$ and we are done.

To show that the inclusion can be proper, let $X = \mathbb{Q}$, $Y = \mathbb{R}$, $f : X \to Y$ the inclusion $\iota : \mathbb{Q} \hookrightarrow \mathbb{R}$, and $E = X = \mathbb{Q}$. Obviously every set is closed as a subset of itself, so $\overline{E} = E$. However, $f(E) = \mathbb{Q} \subset \mathbb{R}$ is dense, and $\overline{f(E)} = \mathbb{R}$. Then $\overline{f(E)} \setminus f(\overline{E}) = \mathbb{R} \setminus \mathbb{Q}$, and hence the inclusion is certainly proper.

4

We have a continuous function $f : X \to \mathbb{R}$. Note that the one point set $\{0\} \subset \mathbb{R}$; indeed, by Rudin Theorem 2.20 finite subsets of arbitrary metric spaces are closed. By Rudin Theorem 4.8 a function is continuous if and only if the inverse image of any closed set is closed. So $Z(f) = f^{-1}(\{0\}) \subset X$ is closed.

If you don't believe that, we can provide essentially the same proof using the previous problem. Let $E = Z(f)$, and note that $f(E) = \{0\}$. Then we have

$$f(\overline{E}) \subset \overline{f(E)} = \overline{\{0\}} = \{0\}$$

Which is to say that $\overline{E} \subset f^{-1}(\{0\}) = E$, i.e. E is closed.