18.100**C**
Fall 2012

Problem Set 4

1. A metric space is called complete if every Cauchy sequence converges. Let (X, d) be a complete metric space, and $f : X \to X$ a map with the following property. There is some $0 \le \lambda < 1$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \le \lambda d(x, y).$$

   Prove that then, there is a point x such that $f(x) = x$. *Hint: you may need the formula* $1 + \lambda + \cdots + \lambda^n = (1 - \lambda^{n+1})/(1 - \lambda)$.
   When writing the answer for this problem, please pay particular attention to completeness of the argument; and to structure, clarity and legibility of writing. **LaTeX is required** *(for this problem only). Your answer will be assessed by the grader for correctness, and then again by the recitation instructor for quality of exposition.* (5 points)

2. Let (x_k) be a convergent sequence in a metric space (X, d). Now permute its terms, forming another sequence $x'_k = x_{g(k)}$, where $g : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto. Show that (x'_k) is convergent, and has the same limit as the original (x_k). Is this still true if we drop the assumption that g should be one-to-one? (2 points)

3. Show that a sequence in $\mathbb{R}^n$ converges if and only if the i-th coordinates of the sequence converge in $\mathbb{R}$ for $i = 1, \ldots, n$. (3 points)

4. Fix some prime p, and let $X = \mathbb{Z}$ with the p-adic metric. Show that the sequence $x_1 = 1$, $x_2 = 1 + p$, $x_3 = 1 + p + p^2$, $\ldots$, is a Cauchy sequence. For $p = 2$, show that this sequence converges. (4 points)

Total: 5 + 2 + 3 + 4 = 14 points.

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