18.100**C**
Fall 2012

Problem Set 8

1. Let (f_n) and (g_n) be two sequences of functions $[a,b] \to \mathbb{R}$, each of which converges uniformly,
$$\lim_n f_n = f, \quad \lim_n g_n = g.$$
Suppose that f and g are bounded. Show that then, $(f_n g_n)$ also converges uniformly to fg. *Please write your solution to this problem out clearly in LaTeX* (3 points).

2. We consider continuous functions $f : [0,1] \to \mathbb{R}$ such that $f(0) = 0$ and $f(1) = 1$. Given such a function f, define another function $\hat{f}$ by
$$\hat{f}(x) = \begin{cases} \frac{1}{4} f(2x) & x < 1/2, \\ \frac{3}{4} f(2x-1) + \frac{1}{4} & x \geq 1/2. \end{cases}$$
Prove that $\hat{f}$ belongs to the same class of functions. Next, prove that $d(\hat{f}, \hat{g}) \leq \frac{3}{4} d(f,g)$, where $d(f,g) = \max\{\|f(x) - g(x)\| \; : \; 0 \leq x \leq 1\}$. Then, prove that there is exactly one continuous function f in our class such that $\hat{f} = f$. (It's fun to try to graph it.) (5 points)

3. Let (f_n) be a sequence of functions $[a,b] \to \mathbb{R}$ such that: (i) $f_n(x) \leq 0$ if n is even, $f_n(x) \geq 0$ if n is odd; (ii) $|f_n(x)| \geq |f_{n+1}(x)|$ for all x; (iii) f_n converges to 0 uniformly. Prove that then,
$$\sum_n f_n$$
is uniformly convergent. (5 points)

Total: 3+5+5 = 13 points.

18.100C Real Analysis
Fall 2012