18.100C Lecture 10 Summary

Theorem 10.1 (Euler). *The series $\sum_p \frac{1}{p}$, where p ranges over all prime numbers, is divergent.*

Absolute convergence of series (of real or complex numbers).

Theorem 10.2. *Absolute convergence implies convergence.*

Theorem 10.3. *Suppose that $\sum_i a_i$ is absolutely convergent, with value s. Then, for every $\epsilon > 0$ there is an N such that the following holds. For every finite subset $I \subset \mathbb{N}$ such that $\{1, \dots, N\} \subset I$, we have*

$$\Big| \sum_{i \in I} a_i - s \Big| < \epsilon.$$

Corollary 10.4. *If $\sum_i a_i$ is absolutely convergent, and $\sum_i a_{\sigma(i)}$ is a reordering (which means that $\sigma : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto), then $\sum_i a_{\sigma(i)}$ is again absolutely convergent, and has the same value.*

This allows us to define absolute convergence for series $\sum_{i \in I} a_i$, where I is any countable set.

Theorem 10.5 (Product theorem for series). *Given series $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$, define their product $\sum_{k=0}^{\infty} c_k$ by setting $c_k = \sum_{i=0}^{k} a_i b_{k-i}$. Suppose that $\sum_j a_i$ is absolutely convergent, and $\sum_j b_j$ convergent. Then $\sum_k c_j$ is again convergent, and*

$$\Big(\sum_i a_i \Big) \cdot \Big(\sum_j b_j \Big) = \sum_k c_k.$$

Root criterion for absolute convergence.