18.100C Lecture 12 Summary

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ a map.

Definition 12.1. *f is continuous (everywhere) if: whenever (x_n) is a sequence in X converging to some point $p \in X$, then $(f(x_n))$ converges to $f(p)$ in Y.*

Definition 12.2. *f is continuous (everywhere) if: for any open subset $V \subset Y$, the preimage $f^{-1}(V) = \{x \in X \ : \ f(x) \in V\}$ is an open subset of X.*

Definition 12.3. *f is continuous (everywhere) if: for all $p \in X$ and all $\epsilon > 0$, there is a $\delta > 0$ such that if $d_X(x, p) < \delta$ then $d_Y(f(x), f(p)) < \epsilon$.*

The "such that..." part can be reformulated as follows: "$f(B_\delta(x)) \subset B_\epsilon(f(x))$". Or as follows: "$B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$".

Definition 12.4. *f is continuous (everywhere) if: for any closed subset $W \subset Y$, the preimage $f^{-1}(W)$ is a closed subset of X.*

Theorem 12.5. *The four definitions above are equivalent.*

Theorem 12.6. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the composition $g \circ f : X \to Z$ is continuous.*

Corollary 12.7. *If $f, g : X \to \mathbb{R}$ (with the usual metric on the real numbers) are continuous, then $f(x) + g(x)$ and $f(x)g(x)$ are continuous.*

Corollary 12.8. *If $f : X \to \mathbb{R}$ is continuous and everywhere nonzero, then $1/f$ is continuous.*

Theorem 12.9. *If $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K) \subset Y$ is compact.*

Corollary 12.10. *If X is a compact metric space and $f : X \to \mathbb{R}$ a continuous function, then f is bounded and has a minimum and maximum.*

Corollary 12.11. *Let X be a compact metric space, and $f : X \to Y$ a map which is continuous, one-to-one, and onto. Then the inverse map $f^{-1} : Y \to X$ (defined by $f(x) = y \Leftrightarrow x = f^{-1}(y)$) is again continuous.*

We return to basic definitions. Let $f : X \to Y$ be a map between metric spaces. Fix a point $p \in X$.

Definition 12.12. *f is continuous at p if: whenever (x_n) is a sequence in X converging to (our particular point) p, then $(f(x_n))$ converges to $f(p)$.*

Definition 12.13. *f is continuous at p if: for all $\epsilon > 0$, there is a $\delta > 0$ such that if $d_X(x, p) < \delta$ then $d_Y(f(x), f(p)) < \epsilon$.*

Theorem 12.14. *The two definitions above are equivalent.*

Definition 12.15. *Let X, Y be metric spaces, $f : X \to Y$ a map, and $p \in X$ a point. We write*

$$\lim_{x \to p} f(x) = q \in Y$$

if the following holds: for all $\epsilon > 0$, there is a $\delta > 0$ such that if $x \neq p$ and $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \epsilon$.

The advantage of this is that it makes sense even if f is defined only on $X \setminus \{p\}$.

Lemma 12.16. *If $f : X \to Y$ satisfies $\lim_{x \to p} f(x) = f(p)$, then it is continuous at p (the converse also holds).*