18.100C Lecture 15 Summary

Theorem 15.1. *Suppose that f and g are functions satisfying $f(g(x)) = x$. Take a point p in the interior of the domain of definition of g, and such that $f(x)$ lies in the interior of the domain of definition of f. Suppose that there is some $\delta > 0$ such that g is increasing on the interval $(p-\delta, p+\delta)$, and that $g'(p)$ exists and is positive (alternatively, g could be strictly decreasing and $g'(p)$ could be negative). Then f' is differentiable at $g(p)$, and*

$$f'(g(p)) = \frac{1}{g'(p)}.$$

Only differentiability needs to be proved; the formula for the derivative then follows from the chain rule.

Example 15.2. *$f(x) = \log(x)$ is differentiable for all $x > 0$, and $f'(x) = 1/x$.*

Example 15.3. *For any natural number n, the function $f(x) = x^{1/n}$ is differentiable for all $x > 0$, and $f'(x) = (1/n)x^{1/n-1}$.*

Definition of higher differentiability. The rest of this lecture is about forms of Taylor's theorem.

Theorem 15.4. *Suppose that f is m times differentiable at p. Then one can write*

$$f(x) = f(p)+(x-p)f'(p)+\frac{(x-p)^2}{2}f''(p)+\cdots+\frac{(x-p)^m}{m!}f^{(m)}(p)+r(x)(x-p)^m,$$

where $\lim_{x\to p} r(x) = 0$.

Equivalently:

Theorem 15.5. *Suppose that f is m times differentiable at p. Then for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x-p| < \delta$, then*

$$\left| f(x) - f(p) - (x-p)f'(p) - \frac{(x-p)^2}{2}f''(p) - \cdots - \frac{(x-p)^m}{m!}f^{(m)}(p) \right| \leq \epsilon |x-p|^m.$$

Theorem 15.6. *Suppose that f is m times differentiable in the (closed) interval bounded by a and b; that $f^{(m)}$ is continuous in the same interval; and that $f^{(m+1)}$ exists at all interior points of that interval. Then*

$$f(b) = f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2}f''(a)+\cdots+\frac{(b-a)^m}{m!}f^{(m)}(a)+\frac{(b-a)^{m+1}}{(m+1)!}f^{(m+1)}(x)$$

for some point x in the interior of the interval bounded by a and b.

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