18.100C Lecture 18 Summary

Definitions of $\mathcal{B}(X)$ (the space of bounded functions), $\mathcal{C}(X)$ (the space of continuous functions) as metric spaces. Also, for $X = [a, b]$, definition of $\mathcal{B}^1(X)$ (the space of functions with bounded derivative) as a metric space.

Theorem 18.1. *$\mathcal{B}(X)$ is a complete metric space.*

Theorem 18.2. *$\mathcal{C}(X)$ is a closed subspace of $\mathcal{B}(X)$, hence itself complete.*

Theorem 18.3. *Take a bounded subset of $\mathcal{B}^1(X)$, consider it as a subset of $\mathcal{C}(X)$, and take its closure with respect to the metric of $\mathcal{C}(X)$. Then that closure is a compact subset of $\mathcal{C}(X)$.*

Uniform approximation by step functions and by piecewise linear functions.

Theorem 18.4. *Let X be a compact metric space. Suppose that $\mathcal{A} \subset C(X)$ is a subset with the following properties: (i) if $f, g \in \mathcal{A}$, then $\max(f, g) \in \mathcal{A}$ and $\min(f, g) \in \mathcal{A}$; (ii) for any two points $x \neq y$ and any real numbers a, b, there is an $f \in \mathcal{A}$ such that $f(x) = a$, $f(y) = b$. Then $\mathcal{A}$ is dense in $C(X)$.*

Theorem 18.5 (Stone-Weierstrass)**.** *Let X be a compact metric space. Suppose that $\mathcal{A} \subset C(X)$ is a subset with the following properties: (i) all constant functions are in $\mathcal{A}$; (ii) if $f, g \in \mathcal{A}$, then $f + g \in \mathcal{A}$; (iii) if $f, g \in \mathcal{A}$, then $f \cdot g \in \mathcal{A}$; (iv) for any two points $x \neq y$, there is an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Then $\mathcal{A}$ is dense in $C(X)$.*

Application: polynomials, trigonometric polynomials.