18.100C Lecture 20 Summary

Definition of Riemann-Stieltjes (RS) integral (of a bounded function, with respect to a nondecreasing function α).

Example 20.1. *Constant functions are always RS integrable, and*

$$\int_a^b c\, d\alpha = c\,(\alpha(b) - \alpha(a)).$$

Example 20.2. *Take some $x_* \in (a, b)$, and define α to be the jump function*

$$\alpha(x) = \begin{cases} 0 & x < x_*, \\ 1 & x \geq x_*. \end{cases}$$

f is RS-integrable with respect to α if $\lim_{x\to x_*-} f(x) = f(x_*)$ holds (in particular, this is true if f is continuous at x_*). In that case,

$$\int_a^b f(x)\, d\alpha = f(x_*).$$

Theorem 20.3. *(i) f is RS-integrable if and only if: for every $\epsilon > 0$, there is a partition P such that*

$$S(f, \alpha, P) - s(f, \alpha, P) < \epsilon.$$

(ii) Suppose that P is a partition as in (i). For each i, take a point $x_i^* \in [x_{i-1}, x_i]$. Then

$$\left| \sum_i f(x_i^*) \Delta\alpha_i - \int_a^b f\, d\alpha \right| < \epsilon.$$

Theorem 20.4. *Continuous functions f are RS-integrable for any α.*

Theorem 20.5. *If (f_n) are RS-integrable with respect to α, and $f_n \to f$ uniformly, then f is RS-integrable for the same α, and*

$$\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha.$$

Theorem 20.6. *(i) If f and g are RS-integrable, then $f + g$ is RS-integrable, and $\int_a^b f + g\, d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha$.*

(ii) If f is RS-integrable and c is a constant, then $c\,f$ is RS-integrable, and $\int_a^b c\,f\, d\alpha = c \int_a^b f\, d\alpha$.

(iii) If f is RS-integrable and $f(x) \geq 0$ for all x, then $\int_a^b f\, d\alpha \geq 0$.