18.100C Lecture 21 Summary

Theorem 21.1. *If f is RS-integrable and ϕ is continuous (on some closed interval containing all values $f(x)$, $x \in [a,b]$), then $\phi(f)$ is RS-integrable (for the same α).*

Corollary 21.2. *If f and g are RS-integrable, then fg is RS-integrable (for the same α).*

Corollary 21.3. *If f is RS-integrable, then $|f|$ is RS-integrable, and $|\int_a^b f\, d\alpha| \leq \int_a^b |f|\, d\alpha$.*

The following is easy:

Theorem 21.4. *Suppose that ϕ is strictly increasing and continuous, and maps $[A,B]$ to $[a,b]$. Then if $f : [a,b] \to \mathbb{R}$ is RS-integrable for some α, $g = f(\phi) : [A,B] \to \mathbb{R}$ is RS-integrable for $\beta = \alpha(\phi)$, and*

$$\int_a^b f\, d\alpha = \int_A^B g\, d\beta.$$

But this is hard:

Theorem 21.5. *Suppose that α is everywhere differentiable, and α' is Riemann-integrable. Let f be a function which is R-S integrable for α. Then $f(x)\alpha'(x)$ is Riemann-integrable, and*

$$\int_a^b f(x)\alpha'(x)\, dx = \int_a^b f\, d\alpha.$$

Together they yield the following form of the substitution rule:

Corollary 21.6. *Suppose that ϕ is strictly increasing, differentiable, maps $[A,B]$ to $[a,b]$, and that ϕ' is Riemann integrable. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Then $f(\phi(x))\phi'(x) : [A,B] \to \mathbb{R}$ is again Riemann integrable, and*

$$\int_A^B f(\phi(x))\phi'(x)\, dx = \int_a^b f(x)\, dx.$$

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