18.100C Lecture 24 Correction

This is a correction to the proof of the following:

Theorem. *Suppose that $h(x)$ is a 2π-periodic function which is differentiable, and $h'(x)$ is continuous. Set $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-ikx}dx$, and define*

$$s_N(h,x) = \sum_{k=-N}^{N} c_k \exp(ikx).$$

Then $s_N(h)$ converges uniformly to f as $N \to \infty$.

We already wrote

$$h(x) - s_N(h,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(h(x) - h(x-t)\Big)\frac{\sin((N+1/2)t)}{\sin(t/2)}\,dt.$$

and

$$\frac{h(x)-h(x-t)}{\sin(t/2)} = \frac{h(x)-h(x-t)}{t}\cdot\frac{t}{\sin(t/2)} = h'(\xi)\frac{t}{\sin(t/2)}.$$

$t/\sin(t/2)$ is bounded on $[-\pi,\pi]$, let's say $|t/\sin(t/2)| \leq D$. Supposing that $|h'(\xi)| \leq C$ everywhere, we have

$$|h(x) - s_N(h,x)| \leq CD \quad \text{for all } N \text{ and all } x.$$

This is not by itself particularly useful, since we are looking for the error to become small as N grows large. However, we now argue as follows.

The Stone-Weierstrass theorem implies that the continuous and 2π-periodic function $h'(x)$ can be approximated uniformly by a trigonometric polynomial. What this means is that for any $\epsilon > 0$ there is some g of the form $g(x) = \sum_{k=-M}^{M} d_k e^{ikx}$ such that $|g(x) - h'(x)| < \epsilon$ everywhere. By integrating $g(x)$ from $-\pi$ to π and comparing that with the integral of $h'(x)$, one sees that $|d_0| < 2\pi\epsilon$. Hence (at the cost of fiddling with ϵ's a little), one may assume that $d_0 = 0$. One can therefore write $g(x) = G'(x)$, where G is again a trigonometric polynomial.

Now

$$s_N(h,x) - h(x) = (s_N(h-G,x) - (h(x) - G(x)) + (s_N(G,x) - G(x)).$$

Since G is a trigonometric polynomial, an explicit computation shows that $s_N(G,x) = G(x)$ if N is large (greater than the constant M appearing previously). On the other hand, $h - G$ has derivative $|h' - g| < \epsilon$, hence by the previous argument (done in class) $|s_N(h-G,x)-(h(x)-G(x))| < D\epsilon$. The two facts together imply that $|s_N(h,x)-h(x)| < D\epsilon$ if N is large. Since D is a universal constant, the desired theorem is proved.