18.100C Lecture 24 Summary

Consider Fourier series

$$a_0 + \sum_{k=1}^{\infty} a_k \sin(kx) + \sum_{k=1}^{\infty} \tilde{a}_k \cos(kx).$$

Lemma 24.1. *If $\sum_k |a_k|$ and $\sum_k |\tilde{a}_k|$ converge, the Fourier series is uniformly convergent for $x \in \mathbb{R}$.*

In that situation, the Fourier series defines a continuous 2π-periodic function $f(x)$.

Lemma 24.2. *If $\sum_k k|a_k|$ and $\sum_k k|\tilde{a}_k|$ converge, the function $f(x)$ defined by the Fourier series is differentiable, and its derivative is*

$$\sum_{k=1}^{\infty} a_k k \cos(kx) - \sum_{k=1}^{\infty} \tilde{a}_k k \sin(kx).$$

It's more convenient to think in terms of complex-valued functions, where Fourier series are

$$\sum_{k \in \mathbb{Z}} c_k \exp(ikt).$$

(convergence here means, say, convergence of the sum from $k = -N$ to $k = N$, as $N \to \infty$). Given any 2π-periodic Riemann integrable function $h(x)$, one defines its Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) e^{-ikx} dx.$$

Theorem 24.3. *Suppose that $h(x)$ is differentiable and $h'(x)$ is continuous. Then the Fourier series $\sum_k c_k \exp(ikx)$ converges uniformly to the original function $h(x)$.*

Theorem 24.4 (Parseval's theorem)**.** *Let $h(x)$ be a 2π-periodic Riemann integrable function. Define its Fourier sums as*

$$s_N(h, x) = \sum_{k=-N}^{N} c_k e^{ikx}.$$

Then we have "average convergence"

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} |h(x) - s_N(h, x)|^2 \, dx = 0$$

18.100C Real Analysis
Fall 2012